Groups, Fields, and Vector Spaces

Homework #2 (2020-2021), Questions

Q1: Putting together groups: Direct products

Let G and H be groups, with elements g, g', etc. in G and h, h', etc. in H, and group operations $\circ_G$ and $\circ_H$. We define the direct product of G and H, $G \times H$, as the set of ordered pairs (g,h), and the group operation $(g,h) \circ (g',h') = (g \circ_G g', h \circ_H h')$.

A. Show that $G \times H$ is a group.

B. Show that the subset S_G consisting of elements in $G \times H$ of the form (g, e_H), (where e_H is the identity for H) is a subgroup of $G \times H$. Is it guaranteed to be a normal subgroup?

C. Let $G = \mathbb{Z}_5$ and $H = \mathbb{Z}_2$. What is the size of $G \times H$? Consider the group D_5 of rotations and reflections of the regular pentagon (i. e., the identity, the four non-trivial rotations by multiples of $2\pi/5$, and the reflections across lines through one vertex and the midpoint of the opposite face). Are $G \times H$ and D_5 the same group? Why or why not?

Q2. Kernels and normal subgroups

The notes showed that if $\varphi : G \to H$ is a homomorphism and $\ker\varphi$ is the set of elements of G for which $\varphi(g) = e_H$, then $\ker\varphi$ is a subgroup of G. Show that $\ker\varphi$ is a normal subgroup.

Q3: Automorphisms

A. What are all the automorphisms of the rational numbers $\mathbb{Q}$ under addition?

B. Are there automorphisms of the real numbers $\mathbb{R}$ (under addition) that do not correspond to automorphisms of $\mathbb{Q}$?

C. What are all the automorphisms of $\mathbb{Q}^n = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ under addition? (See Q1 for definition of the direct product $\times$)

D. What are all the automorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_2$?