Groups, Fields, and Vector Spaces (and a bit from Linear Transformations)

Homework #4 (2020-2021), Answers

Q1: Tensor products in 2 dimensions
The general setup is taken from the section of the notes on the effect of coordinate changes on tensor products: two vector spaces V and W, a coordinate change in V corresponding to an invertible $m\times m$ matrix A (with $v=Av'$, and a coordinate change in W corresponding to an invertible $n\times n$ matrix B (or a with $w=Bw'$). We considered a tensor product $q\in V\otimes W$ expressed in coordinates, as $q=\sum_{i=1}^{m}\sum_{j=1}^{n}q_{i,j}(v_i\otimes w_j)$. We then showed that in the new coordinate system, $q=\sum_{k=1}^{m}\sum_{m=1}^{n}q_{k,l}{}'(v_k\otimes w_l)$, where the coefficients $q'_{k,l}$ of $v'_k\otimes w'_l$ are given by $q'_{k,l}=\sum_{i=1}^{m}\sum_{j=1}^{n}q_{i,j}A_{i,k}B_{j,l}$. Here, we specialize this to the case $V=W$, $A=B$, and $m=n=2$.

A. Write out $q'_{k,l}=\sum_{i=1}^{m}\sum_{j=1}^{n}q_{i,j}A_{i,k}B_{j,l}$ explicitly for this special case (without summation notation).

$$q'_{k,l}=q_{1,1}A_{1,k}A_{1,l}+q_{1,2}A_{1,k}A_{2,l}+q_{2,1}A_{2,k}A_{1,l}+q_{2,2}A_{2,k}A_{2,l}$$

B. Show that if $q_{i,j}=q_{j,i}$, then $q'_{k,l}=q'_{l,k}$. Note that this means we found a subspace of $V\otimes V$ that is invariant under all coordinate changes. What is its dimension?

The conditions and the conclusions automatically holds "on the diagonal", i.e., for $i=j$ and $k=l$. Off the diagonal: let $q_{1,2}=q_{2,1}=q_0$. Then

$q'_{1,2}=q_{1,1}A_{1,1}A_{1,2}+q_0\left(A_{1,1}A_{2,2}+A_{2,1}A_{1,2}\right)+q_{2,2}A_{2,1}A_{2,2}$, while

$q'_{2,1}=q_{1,1}A_{1,2}A_{1,1}+q_0\left(A_{1,2}A_{2,1}+A_{2,2}A_{1,1}\right)+q_{2,2}A_{2,2}A_{2,1}$.

These are equal, term by term.

The dimension of the subspace of $V\otimes V$ that satisfies $q_{i,j}=q_{j,i}$ is 3: $q_{1,1}$ and $q_{1,2}$ are free to vary, and the other terms are constrained by $q_{1,2}=q_{2,1}=q_0$.

C. *Show that if $q_{i,j}=-q_{j,i}$, then $q'_{k,l}=-q'_{l,k}$. Note that this means we found another subspace of $V\otimes V$ that is invariant under all coordinate changes. What is its dimension?*

On the diagonal, the conditions require that $q_{1,1}=0$ and $q_{2,2}$=0. Off the diagonal: let $q_{1,2}=q_d$, and $q_{2,1}=-q_d$. Then

$q'_{1,1}=q_d\left(A_{1,1}A_{2,1}-A_{2,1}A_{1,1}\right)=0$

$q'_{1,2}=q_d\left(A_{1,1}A_{2,2}-A_{2,1}A_{1,2}\right)$,

$q'_{2,1} = q_d \left(A_{1,2}A_{2,1} - A_{2,2}A_{1,1} \right) = -q'_{1,2}$,
$q'_{2,2} = q_d \left(A_{1,2}A_{2,2} - A_{2,2}A_{1,2} \right) = 0$.

There's only one free parameter, so the dimension is 1. The coordinate change by A in V has been mapped into a linear transformation in this subspace, namely, multiplication by $A_{1,2}A_{2,1} - A_{2,2}A_{1,1}$.

Q2: Determinants of some transformations from first principles

Setup: V is a vector space of dimension m, and A is a linear transformation in $Hom(V,V)$. Further assume that A has m linearly-independent eigenvectors w_j, with $Aw_j = \lambda_j w_j$.

A. Find $\det(A)$ *from the its definition as* $\dfrac{anti((Av)^{\otimes m})}{anti(v^{\otimes m})}$.

We take $v^{\otimes m} = w_1 \otimes w_2 \otimes \cdots \otimes w_m$. Then

$$Av^{\otimes m} = Aw_1 \otimes Aw_2 \otimes \cdots \otimes Aw_m = \lambda_1 w_1 \otimes \lambda_2 w_2 \otimes \cdots \otimes \lambda_m w_m = (\lambda_1\lambda_2 \cdots \lambda_m)\left(w_1 \otimes w_2 \otimes \cdots \otimes w_m\right) = (\lambda_1\lambda_2 \cdots \lambda_m)v^{\otimes m}.$$

This is also true for any permutation of the w_j's. So

$$\det(A) = \frac{anti((Av)^{\otimes m})}{anti(v^{\otimes m})} = \lambda_1\lambda_2 \cdots \lambda_m.$$

B. With the above setup, find $\det(A \otimes A)$, *where* $A \otimes A$ *is the linear transformation of* $V \otimes V$ *defined by* $(A \otimes A)(v \otimes v') = Av \otimes Av'$.

$V \otimes V$ has dimension m^2. We reduce this to Q2A by finding a set of m^2 linearly-independent eigenvectors for $A \otimes A$: the elementary tensors $w_j \otimes w_k$. They are linearly-independent, because if not, we could use the rule for combining elementary tensor products to deduce linear dependence among the w_j's. Their eigenvalues are given by $(A \otimes A)(w_j \otimes w_k) = (\lambda_j w_j \otimes \lambda_k w_k) = \lambda_j \lambda_k (w_j \otimes w_k)$. So the product of the eigenvalues, which is (by Q2A) is $\det(A \otimes A)$, is $(\lambda_1\lambda_1)(\lambda_1\lambda_2)\cdots(\lambda_1\lambda_m)(\lambda_2\lambda_1)\cdots(\lambda_2\lambda_m)\cdots(\lambda_m\lambda_1)(\lambda_m\lambda_2)\cdots(\lambda_m\lambda_m)$. This is $\left(\lambda_1\lambda_2 \cdots \lambda_m\right)^{2m}$, since each λ occurs m times in the first and second position of this product.