Linear Systems, Black Boxes, and Beyond

Homework #3 (2020-2021), Questions

Q1: Spectra of some other renewal processes

This is a computational exercise about non-Poisson renewal processes. A "gamma process" of order m (here, $m>0$) and rate λ is a renewal process whose renewal density is $g_m(t;\lambda)=\frac{t^{m-1}(\lambda m)^m}{\Gamma(m)}e^{-t\lambda m}$. ($\Gamma(m)$ is the gamma-function, $\Gamma(m)=\int_0^\infty u^{m-1}e^{-u}du$, and $\Gamma(m)=(m-1)!$ for $m=1,2,3,\ldots$). Note that g_m is properly normalized: $\int_0^\infty g_m(t;\lambda)dt=\frac{1}{\Gamma(m)}\int_0^\infty t^{m-1}(\lambda m)^m e^{-t\lambda m}dt=\frac{1}{\Gamma(m)}\int_0^\infty u^{m-1}e^{-u}du=1$, with second step using $u=t\lambda m$. For integers $m\geq 1$, a gamma-process can be derived from a Poisson process of rate λm by taking every m th event. We don't show this here; see supplementary material below.

What this means is that the renewal density for $g_m(t)$ is the m -fold convolution of the renewal density of a Poisson process of rate $m\lambda$, with itself. Since convolution in the time domain is multiplication in the frequency domain, it follows that $\tilde{g}_m(\omega;\lambda)=\left(\frac{1}{1+\frac{i\omega}{m\lambda}}\right)^m$ -- as shown in the supplementary material below.

Using this as a starting point:

A: Plot the renewal density of a gamma process of order m, i.e., $g_m(t;\lambda)=\frac{t^{m-1}(\lambda m)^m}{\Gamma(m)}e^{-t\lambda m}$, for a few values of m and λ.
B. Plot the corresponding power spectra.
C. How do you interpret the behavior of the power spectrum as $\omega\to 0$ and as $\omega\to\infty$?
D. For what value of m does the power spectrum first have a peak at a nonzero frequency?

Q2-Q4 concern the "global coherence, which is a kind of generalization of pairwise coherence. See Cimenser et al., "Tracking brain states under general anesthesia using global coherence analysis", PNAS 108, 8832-8837.

Q2: Cross-spectral matrix and global coherence: definition and basic properties

Say we have a set $X_1,X_2,\ldots,X_N$ of random signals. Let $P_{X_j,X_k}(\omega)$ is the cross-spectrum of X_j and X_k. The cross-spectral matrix $M(\omega)$ is defined as the matrix whose elements $M_{j,k}(\omega)=P_{X_j,X_k}(\omega)$. The global coherence at the frequency ω is defined as the ratio of the largest eigenvalue of $M(\omega)$ to the sum of its eigenvalues.

A. Is $M(\omega)$ self-adjoint?

B. Part A means that the eigenvalues of $M(\omega)$ are real. Here we show that they also must be non-negative.

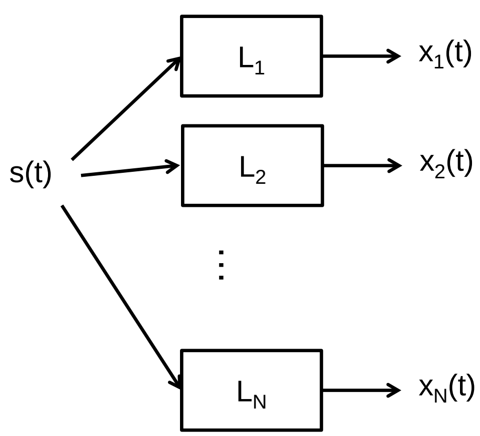
First, show that if a matrix A has the property that z^*Az is real and non-negative for all vectors $z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$ (where z^* is the conjugate transpose of z), then all eigenvalues of A are non-negative.
Then, using the definition of the cross-spectrum in terms of Fourier estimates,

$P_{X,Y}(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle F(x,\omega,T,0)\overline{F(y,\omega,T,0)} \right\rangle$, show that for any vector $z(\omega) = \begin{pmatrix} z_1(\omega) \\ z_2(\omega) \\ \vdots \\ z_N(\omega) \end{pmatrix}$, that $z^*(\omega)M(\omega)z(\omega) \geq 0$.

C. What is the smallest possible value of the global coherence of N signals?
$1/N$: Since the eigenvalues are non-negative and sum to the trace, the largest eigenvalue must be at least $1/N$ th of the trace.

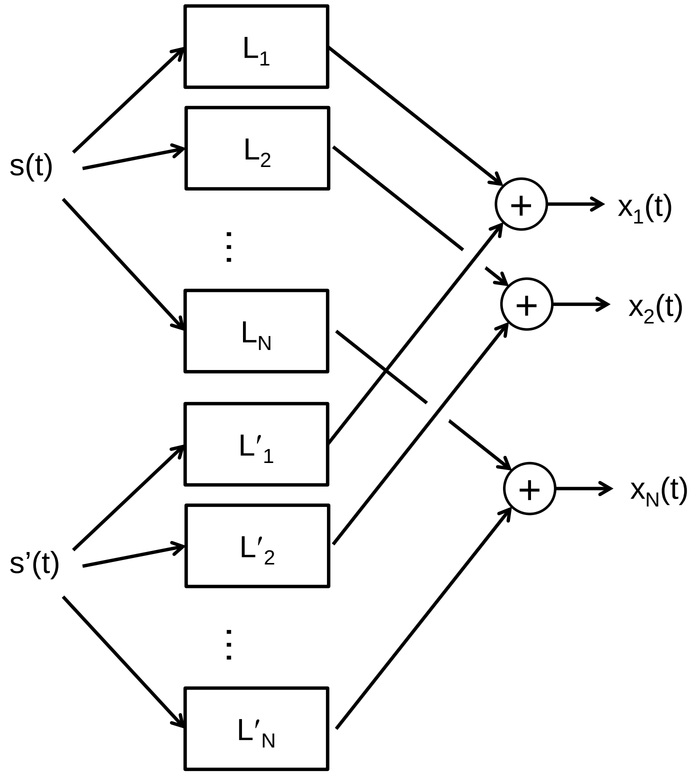
Q3: Global coherence: a single, common noise source

A. Consider the following system, in which each of the signals X_j are generated by a separate linear filter L_j acting on the same noise input $s(t)$, whose power spectrum is $P_S(\omega)$.. Determine $P_{X_j,X_k}(\omega)$ in terms of and $P_S(\omega)$ and the transfer functions $\tilde{L}_j(\omega)$ and $\tilde{L}_k(\omega)$ of L_j and L_k.



B. Show that the vector $v(\omega) = \begin{pmatrix} \tilde{L}_1(\omega) \\ \tilde{L}_2(\omega) \\ \vdots \\ \tilde{L}_N(\omega) \end{pmatrix}$ is an eigenvector of the cross-spectral matrix $M(\omega)$, and find its corresponding eigenvalue.

Q4. Now consider the following system, where $s(t)$ and $s'(t)$ are independent noises, with power spectra $P_{X_j,X_k}(\omega)$ $P_S(\omega)$ and $P_{S'}(\omega)$; the rest of the set-up is as above.



A. Determine $P_{X_j,X_k}(\omega)$

B. Show that the range of the cross-spectral matrix $M(\omega)$ is of dimension at most 2.

Supplementary material for Q1 about gamma processes

Here we determine the Fourier transform of the renewal density of a gamma process. We do this by finding the renewal density of a gamma process of order m and rate λ / m (rather than rate λ), since – as the calculation will show – this is the m-fold convolution of the renewal density of a Poisson process of rate λ. That is, we determine the Fourier transform of $s_m(t;\lambda) = g_m(t;\lambda / m) = \frac{t^{m-1}\lambda^m}{\Gamma(m)} e^{-t\lambda}$.

We do this via a method, "generating functions", that is widely useful, produces the answer for all m at once. The idea is that we look at $S(t,y;\lambda) = \sum_{m=1}^{\infty} y^{m-1} s_m(t;\lambda)$, and compute its Fourier transform. Since $\tilde{S}(\omega,y;\lambda) = \int_0^{\infty} S(t,y;\lambda) e^{-i\omega t} dt = \int_0^{\infty} \sum_{m=1}^{\infty} y^{m-1} s_m(t;\lambda) dt = \sum_{m=1}^{\infty} y^{m-1} \tilde{s}_m(\omega;\lambda)$, we can then pull out the terms involving y^{m-1} in $\tilde{S}(\omega,y;\lambda)$ to get the Fourier transform $\tilde{s}_m(\omega;\lambda)$ of $s_m(t;\lambda)$.

The generating-function method works because $S(t,y;\lambda)$ has a nice form:

$$S(t,y;\lambda) = \sum_{m=1}^{\infty} y^{m-1} s_m(t;\lambda) = \sum_{m=1}^{\infty} y^{m-1} \frac{t^{m-1}\lambda^m}{\Gamma(m)} e^{-t\lambda} = \lambda e^{-t\lambda} \sum_{m=1}^{\infty} \frac{y^{m-1} t^{m-1} \lambda^{m-1}}{\Gamma(m)}$$
$$= \lambda e^{-t\lambda} \sum_{n=0}^{\infty} \frac{(yt\lambda)^n}{\Gamma(n+1)} = \lambda e^{-t\lambda} \sum_{n=0}^{\infty} \frac{(yt\lambda)^n}{n!} = \lambda e^{-t\lambda + yt\lambda}.$$

So,

$$\tilde{S}(\omega,y;\lambda) = \int_0^{\infty} S(t,y;\lambda) e^{-i\omega t} dt = \int_0^{\infty} \lambda e^{-t\lambda + yt\lambda} e^{-i\omega t} dt$$
$$= \lambda \frac{1}{-\lambda + y\lambda - i\omega} e^{-t\lambda + yt\lambda} e^{-i\omega t} \Big|_0^{\infty} = \lambda \frac{1}{\lambda + i\omega - y\lambda} = \frac{\lambda}{\lambda + i\omega} \frac{1}{1 - \frac{\lambda y}{\lambda + i\omega}}.$$

Note that the final expression is of the form $a\frac{1}{1-ry}$, the sum of a geometric series whose nth term is $ar^n y^n$.

So the term involving y^{m-1} is $\frac{\lambda}{\lambda + i\omega} \left(\frac{\lambda}{\lambda + i\omega} \right)^{m-1} y^{m-1} = \left(\frac{\lambda}{\lambda + i\omega} \right)^m y^{m-1}$. So, the Fourier transform $\tilde{s}_m(\omega;\lambda)$ of $s_m(t;\lambda)$ is the coefficient of y^{m-1} in this term, namely, $\left(\frac{\lambda}{\lambda + i\omega} \right)^m$. So $\tilde{s}_m(\omega;\lambda) = \left(\frac{\lambda}{\lambda + i\omega} \right)^m = \left(\frac{1}{1 + \frac{i\omega}{\lambda}} \right)^m$, corresponding to the m-fold convolution of the Poisson renewal density with itself.

Finally,

$$g_m(t;\lambda) = s_m(t;m\lambda), \text{ so } \tilde{g}_m(\omega;\lambda) = \tilde{s}_m(\omega;m\lambda) = \left(\frac{1}{1 + \frac{i\omega}{m\lambda}} \right)^m.$$