Linear Transformations and Group Representations

Homework #1 (2020-2021), Questions

Q1: Eigenvalues and eigenvectors of a rotation matrix. Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Use the characteristic equation to find its eigenvalues, and then find its eigenvectors.

Q2: Eigenvectors and eigenvalues of permutation matrices.

A. Cyclic permutation matrices. Let $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. Note that $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_1 \end{pmatrix}$, i.e., Ax permutes the entries of the vector x. Use this to write the five (very simple) equations corresponding to $Ax = \lambda x$, and thereby find the eigenvalues and eigenvectors of A.

B. More general permutation matrices. Same as part A, but with $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.