Linear Transformations and Group Representations

Homework #2 (2020-2021), Answers

Q1: Shared eigenvectors and commuting operators.

We are considering the vector space V *of smooth complex-valued functions of time. In the notes, we considered the family of time-translation operator* D_T*, which acts on* $f \in V$ *by* $\left(D_T f\right)(t) = f(t+T)$*, and we showed that for any* D_T*, the eigenvectors are given by* $v_\omega(t) = e^{i\omega T}$*, and they form a basis. For each of the following operators, write its action in terms of the basis set* $\left\{v_\omega\right\}$*. Use this representation to determine whether the operator commutes with* D_T*.*

A. L_{deriv} *(take the derivative), defined by* $\left(L_{deriv} f\right)(t) = \dfrac{df}{dt}$.

$\left(L_{deriv} v_\omega\right)(t) = \dfrac{d}{dt}\nu_\omega(t) = \dfrac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t} = i\omega\nu_\omega(t)$, so $L_{deriv}v_\omega = i\omega\nu_\omega$. This is diagonal in the basis set $\left\{v_\omega\right\}$, so L_{deriv} commutes with all the D_T.

B. L_{boxcar} *(boxcar smoothing), defined by* $\left(L_{boxcar} f\right)(t) = \dfrac{1}{2h}\displaystyle\int_{-h}^{h} f(t+\tau)d\tau$.

$$\left(L_{boxcar} v_\omega\right)(t) = \frac{1}{2h}\int_{-h}^{h} v_\omega(t+\tau)d\tau = \frac{1}{2h}\int_{-h}^{h} e^{i\omega(t+\tau)}d\tau = e^{i\omega t}\frac{1}{2h}\int_{-h}^{h} e^{i\omega\tau}d\tau =$$
$$e^{i\omega t}\frac{1}{2h}\frac{1}{i\omega}e^{i\omega\tau}\Big|_{-h}^{h} = e^{i\omega t}\frac{1}{2i\omega h}\left(e^{i\omega h} - e^{-i\omega h}\right) = e^{i\omega t}\frac{\sin\omega h}{\omega h} = \frac{\sin\omega h}{\omega h}v_\omega(t)$$

, so $L_{boxcar}v_\omega = \dfrac{\sin\omega h}{\omega h}\nu_\omega$. This is diagonal in the basis set $\left\{v_\omega\right\}$, so L_{boxcar} commutes with all the D_T.

C. L_{even} *(make even-symmetric), defined by* $\left(L_{even} f\right)(t) = \dfrac{1}{2}\left(f(t) + f(-t)\right)$.

$\left(L_{even} v_\omega\right)(t) = \dfrac{1}{2}\left(\nu_\omega(t) + v_\omega(-t)\right) = \dfrac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) = \dfrac{1}{2}\nu_\omega(t) + \dfrac{1}{2}\nu_{-\omega}(t)$, so $L_{even}v_\omega = \dfrac{1}{2}\left(\nu_\omega + \nu_{-\omega}\right)$. This is not diagonal in the basis set $\left\{v_\omega\right\}$, so L_{even} may not commute with all the D_T. In fact,

$\left(D_T\left(L_{even} f\right)\right)(t) = \dfrac{1}{2}\left(f(t+T) + f(-t-T)\right)$, while $\left(L_{even}\left(D_T f\right)\right)(t) = \dfrac{1}{2}\left(f(t+T) + f(-t+T)\right)$.

D. L_{odd} *(make odd-symmetric), defined by* $\left(L_{odd} f\right)(t) = \dfrac{1}{2}\left(f(t) - f(-t)\right)$.

$\left(L_{odd} v_\omega\right)(t) = \dfrac{1}{2}\left(\nu_\omega(t) - v_w(-t)\right) = \dfrac{1}{2}\left(e^{i\omega t} - e^{-i\omega t}\right) = \dfrac{1}{2}\nu_\omega(t) - \dfrac{1}{2}\nu_{-\omega}(t)$, so $L_{odd}v_\omega = \dfrac{1}{2}\left(\nu_\omega - \nu_{-\omega}\right)$. This is not diagonal in the basis set $\left\{v_\omega\right\}$, so L_{odd} may not commute with all the D_T. In fact,

$\left(D_T\left(L_{odd} f\right)\right)(t) = \dfrac{1}{2}\left(f(t+T) - f(-t-T)\right)$, while $\left(L_{odd}\left(D_T f\right)\right)(t) = \dfrac{1}{2}\left(f(t+T) - f(-t+T)\right)$.

E. Which of the above are projections?

L_{deriv} and L_{boxcar} cannot be projections, since their eigenvalues for v_ω of $i\omega\nu_\omega$ and $\frac{\sin\omega h}{\omega h}$, which need not be equal to 0 or 1.

For L_{even} and L_{odd}: Since $L_{even}v_\omega = \frac{1}{2}(\nu_\omega + \nu_{-\omega})$ and $L_{even}v_{-\omega} = \frac{1}{2}(\nu_{-\omega} + \nu_\omega)$, we can write the action of L_{even} in the subspace spanned by v_ω and v_ω as $L_{even}\begin{pmatrix} v_\omega \\ v_{-\omega} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} v_\omega \\ v_{-\omega} \end{pmatrix}$. It is easy to verify that the matrix $M_{even} = \frac{1}{2}\begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix}$ satisfies $M_{even}{}^2 = M_{even}$ and is self-adjoint (it is equal to its transpose-conjugate).

Similarly,

$L_{odd}\begin{pmatrix} v_\omega \\ v_{-\omega} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}\begin{pmatrix} v_\omega \\ v_{-\omega} \end{pmatrix}$, and $M_{odd} = \frac{1}{2}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$ also has these properties.

Q2. A five-dimensional space associated with symmetric tensors in 3 dimensions.

This has a long setup and is a bit heavy on algebra, but I think it's worthwhile to how the machinery works, and, it will also serve as an example later on for how group representations work when the set of transformations is not commutative. This question shows how the rotations act on the symmetric tensors in 3-space: that there is a one-dimensional subspace that is invariant when the 3-d coordinates are rotated, and it will demonstrate the way that the coordinate rotations act on the other 5 dimensions.

The setup is the tensor products q of elements in an n-dimensional vector space V, in which we've chosen basis vectors $x_1, x_2, \ldots, x_n$. As we've seen, the tensor-product space $V \otimes V$ has n^2 dimensions as it has a basis consisting of the $x_i \otimes x_j$, and typical members of $V \otimes V$ can be written as $q = \sum_{i,j=1}^{n} q_{i,j} x_i \otimes x_j$. We've seen that an isomorphism A of V yields an isomorphism Ψ_A in $V \otimes V$, namely, $\Psi_A(x_i \otimes x_j) = Ax_i \otimes Ax_j$. We've also seen that Ψ_A maps the symmetric part of $V \otimes V$ into itself, showing that it is an intrinsic aspect of the structure of $V \otimes V$. A basis for the symmetric part can be found by symmetrizing the basis for $V \otimes V$, and consists of $x_i \otimes x_i$ and $\frac{1}{2}(x_i \otimes x_j + x_j \otimes x_i)$ for $i \neq j$. This also showed that the symmetric part of $V \otimes V$ has $\frac{n(n+1)}{2}$ dimensions.

As is suggested by the example of the diffusion tensor, it is convenient to think of symmetric tensor products as quadratics, i.e., to think of $x_i \otimes x_i$ as x_i^2, and to think of $\frac{1}{2}(x_i \otimes x_j + x_j \otimes x_i)$ as $x_i x_j$ -- because they transform in the same way.

We now specialize the above picture two ways. First – and this is just to make things more concrete – we set $n = 3$. But also, we only consider the isomorphisms R of V that preserve the length: $\Psi_R(x_1^2 + x_2^2 + x_3^2) = x_1^2 + x_2^2 + x_3^2$. This is equivalent to saying that V has Hilbert space structure and R preserves the dot-product: $\langle Rx, Ry \rangle = \langle x, y \rangle$. We now have a scenario in which Ψ_R acts in a 6-dimensional space (the quadratic polynomials in x_1, x_2, and x_3), and preserves a one-dimensional subspace of it, namely, scalar multiples of $x_1^2 + x_2^2 + x_3^2$. So, complementary to this one-dimensional subspace, there must be a 5-dimensional subspace in which Ψ_R acts non-trivially – and the goal here is examine this action.

We choose the following basis for $sym(V\otimes V)$ *:*

$$r = x_1^2 + x_2^2 + x_3^2$$
$$s = \frac{1}{\sqrt{3}}\left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right)$$
$$t = \frac{1}{2}\left(x_2^2 - x_3^2\right)$$

. Note that this is a basis, as $\{r,s,t\}$ *allow for any linear combination of the* x_i^2*,*

$$u = x_1x_2$$
$$v = x_1x_3$$
$$w = x_2x_3$$

while $\{u,v,w\}$ *allow for any linear combination of the cross-terms. Further,* $\Psi_R(r) = r$ *. Side note: the seemingly strange coefficients in front of* s *and* t *are determined so that* $\{s,t,u,v,w\}$ *have the same mean-squared value when averaged over a sphere.*

We'll now examine how the rotations act on the five-dimensional space spanned by $\{s,t,u,v,w\}$*. Since any rotation can be generated by composing rotations around the three coordinate axes, it suffices to consider the following three rotations:*

$R_1(\theta)$*, given in coordinates by* $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$*,*

$R_2(\theta)$*, given in coordinates by* $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$*, and*

$R_3(\theta)$*, given in coordinates by* $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$*.*

For each of the above, find the coordinate transformation that relates $\{s',t',u',v',w'\}$ *to* $\{s,t,u,v,w\}$*.*

For $R_1(\theta)$:

$$s' = \frac{1}{\sqrt{3}}\left(x_1'^2 - \frac{1}{2}x_2'^2 - \frac{1}{2}x_3'^2\right) = \frac{1}{\sqrt{3}}\left(x_1^2 - \frac{1}{2}\left(x_2\cos\theta + x_3\sin\theta\right)^2 - \frac{1}{2}\left(-x_2\sin\theta + x_3\cos\theta\right)^2\right) = \frac{1}{\sqrt{3}}\left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right) = s$$

$$t' = \frac{1}{2}\left(x_2'^2 - x_3'^2\right) = \frac{1}{2}\left(\left(x_2\cos\theta + x_3\sin\theta\right)^2 - \left(-x_2\sin\theta + x_3\cos\theta\right)^2\right) =$$

$$\frac{1}{2}\left(x_2^2\left(\cos^2\theta - \sin^2\theta\right) - x_3^2\left(\cos^2\theta - \sin^2\theta\right)^2 + 4x_2x_3\cos\theta\sin\theta\right) = \frac{1}{2}\left(x_2^2 - x_3^2\right)\cos 2\theta + \left(x_2x_3\right)\sin 2\theta$$

$$= t\cos 2\theta + w\sin 2\theta$$

$$u' = x_1'x_2' = x_1\left(x_2\cos\theta + x_3\sin\theta\right) = x_1x_2\cos\theta + x_1x_3\sin\theta = u\cos\theta + v\sin\theta$$

$$v' = x_1'x_3' = x_1\left(-x_2\sin\theta + x_3\cos\theta\right) = -x_1x_2\sin\theta + x_1x_3\cos\theta = -u\sin\theta + v\cos\theta$$

$$w' = x_2'x_3' = \left(x_2\cos\theta + x_3\sin\theta\right)\left(-x_2\sin\theta + x_3\cos\theta\right) =$$

$$x_2^2\left(-\cos\theta\sin\theta\right) + x_3^2\left(\cos\theta\sin\theta\right) + x_2x_3\left(\cos^2\theta - \sin^2\theta\right) = \qquad .$$

$$-\frac{1}{2}(x_2^2 - x_3^2)\left(2\cos\theta\sin\theta\right) + x_2x_3\left(\cos^2\theta - \sin^2\theta\right) = -t\sin 2\theta + w\cos 2\theta$$

So, in the 5-d space, $R_1(\theta)$ becomes:

$$\begin{pmatrix} s' \\ t' \\ u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos 2\theta & 0 & 0 & \sin 2\theta \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & -\sin 2\theta & 0 & 0 & \cos 2\theta \end{pmatrix} \begin{pmatrix} s \\ t \\ u \\ v \\ w \end{pmatrix}.$$

For $R_2(\theta)$:

$$s' = \frac{1}{\sqrt{3}}\left(x_1'^2 - \frac{1}{2}x_2'^2 - \frac{1}{2}x_3'^2\right) = \frac{1}{\sqrt{3}}\left(\left(x_1\cos\theta - x_3\sin\theta\right)^2 - \frac{1}{2}x_2^2 - \frac{1}{2}\left(x_1\sin\theta + x_3\cos\theta\right)^2\right) =$$

$$\frac{1}{\sqrt{3}}\left(x_1^2\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) - \frac{1}{2}x_2^2 + x_3^2\left(\sin^2\theta - \frac{1}{2}\cos^2\theta\right) - 3x_1x_3\cos\theta\sin\theta\right) =$$

$$\frac{1}{\sqrt{3}}\left((x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2)\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) + \frac{1}{2}(x_2^2 + x_3^2)\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) - \frac{1}{2}x_2^2 + x_3^2\left(\sin^2\theta - \frac{1}{2}\cos^2\theta\right) - 3x_1x_3\cos\theta\sin\theta\right)$$

$$s\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{2}(x_2^2 + x_3^2)\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) - \frac{1}{2}x_2^2 + x_3^2\left(\sin^2\theta - \frac{1}{2}\cos^2\theta\right) - 3x_1x_3\cos\theta\sin\theta\right) =$$

$$s\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{2}x_2^2\left(\cos^2\theta - 1 - \frac{1}{2}\sin^2\theta\right) + \frac{1}{2}x_3^2\left(\cos^2\theta - \frac{1}{2}\sin^2\theta + 2\sin^2\theta - \cos^2\theta\right) - 3x_1x_3\cos\theta\sin\theta\right) =$$

$$s\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) + \frac{1}{\sqrt{3}}\left(-\frac{3}{4}x_2^2\sin^2\theta + \frac{3}{4}x_3^2\sin^2\theta - 3x_1x_3\cos\theta\sin\theta\right) =$$

$$s\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) - \frac{\sqrt{3}}{2}t\sin^2\theta - \frac{\sqrt{3}}{2}v\sin 2\theta = s\left(\frac{1}{4} + \frac{3}{4}\cos 2\theta\right) - t\frac{\sqrt{3}}{4}(1 - \cos 2\theta) - \frac{\sqrt{3}}{2}v\sin 2\theta$$

$$t' = \frac{1}{2}\left(x_2'^2 - x_3'^2\right) = \frac{1}{2}\left(x_2^2 - \left(x_1 \sin\theta + x_3 \cos\theta\right)^2\right) = \frac{1}{2}\left(x_2^2 - \left(x_1 \sin\theta + x_3 \cos\theta\right)^2\right) =$$
$$-\frac{1}{2}x_1^2 \sin^2\theta + \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2 \cos^2\theta - x_1 x_3 \cos\theta \sin\theta =$$
$$-\frac{1}{2}\left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right)\sin^2\theta + \frac{1}{2}x_2^2\left(1 - \frac{1}{2}\sin^2\theta\right) - \frac{1}{2}x_3^2\left(\cos^2\theta + \frac{1}{2}\sin^2\theta\right) - x_1 x_3 \cos\theta \sin\theta =$$
$$-s\frac{\sqrt{3}}{2}\sin^2\theta + \frac{t}{2}\left(1 + \cos^2\theta\right) - v\cos\theta\sin\theta =$$
$$-s\frac{\sqrt{3}}{4}(1 - \cos 2\theta) + \frac{t}{4}(3 + \cos 2\theta) - \frac{v}{2}\sin 2\theta$$
$$u' = x_1' x_2' = \left(x_1 \cos\theta - x_3 \sin\theta\right) x_2 = x_1 x_2 \cos\theta - x_2 x_3 \sin\theta = u\cos\theta - w\sin\theta$$
$$v' = x_1' x_3' = \left(x_1 \cos\theta - x_3 \sin\theta\right)\left(x_1 \sin\theta + x_3 \cos\theta\right) =$$
$$x_1^2 \cos\theta \sin\theta - x_3^2 \cos\theta \sin\theta + x_1 x_3\left(\cos^2\theta - \sin^2\theta\right) = \left(x_1^2 - x_3^2\right)\cos\theta\sin\theta + x_1 x_3 \cos 2\theta$$
$$= \left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right)\cos\theta\sin\theta + \frac{1}{2}(x_2^2 - x_3^2)\cos\theta\sin\theta + x_1 x_3 \cos 2\theta$$
$$= s\frac{\sqrt{3}}{2}\sin 2\theta + t\frac{1}{2}\sin 2\theta + v\cos 2\theta$$
$$w' = x_2' x_3' = x_2\left(x_1 \sin\theta + x_3 \cos\theta\right) = x_1 x_2 \sin\theta + x_2 x_3 \cos\theta = u\sin\theta + w\cos\theta\,.$$

So, in the 5-d space, $R_2(\theta)$ becomes:

$$\begin{pmatrix} s' \\ t' \\ u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{3}{4}\cos 2\theta & -\frac{\sqrt{3}}{4}\left(1 - \cos 2\theta\right) & 0 & -\frac{\sqrt{3}}{2}\sin 2\theta & 0 \\ -\frac{\sqrt{3}}{4}\left(1 - \cos 2\theta\right) & \frac{3}{4} + \frac{1}{4}\cos 2\theta & 0 & -\frac{1}{2}\sin 2\theta & 0 \\ 0 & 0 & \cos\theta & 0 & -\sin\theta \\ \frac{\sqrt{3}}{2}\sin 2\theta & \frac{1}{2}\sin 2\theta & 0 & \cos 2\theta & 0 \\ 0 & 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} s \\ t \\ u \\ v \\ w \end{pmatrix}.$$

Computing $R_3(\theta)$ follows the same lines as $R_2(\theta)$, with t exchanged for $-t$ and u and v exchanged. So, in the 5-d space, $R_3(\theta)$ becomes:

$$\begin{pmatrix} s' \\ t' \\ u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{3}{4}\cos 2\theta & \frac{\sqrt{3}}{4}\left(1 - \cos 2\theta\right) & -\frac{\sqrt{3}}{2}\sin 2\theta & 0 & 0 \\ \frac{\sqrt{3}}{4}\left(1 - \cos 2\theta\right) & \frac{3}{4} + \frac{1}{4}\cos 2\theta & \frac{1}{2}\sin 2\theta & 0 & 0 \\ \frac{\sqrt{3}}{2}\sin 2\theta & -\frac{1}{2}\sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} s \\ t \\ u \\ v \\ w \end{pmatrix}.$$

One can check that all of these 5-d matrices are rotations (they have orthonormal rows and columns), and that the all have the same trace, namely $1+2\cos\theta+2\cos 2\theta$.

Q3: Projections onto subspaces of dimension greater than 1.
The notes asserted that for a linear transformation B, it follows that $P_B = B(B^*B)^{-1}B^*$ is a projection. Here we show it.

A. Show that $P_B{}^2 = P_B$.
$P_B{}^2 = \left(B(B^*B)^{-1}B^*\right)\left(B(B^*B)^{-1}B^*\right) = B(B^*B)^{-1}(B^*B)(B^*B)^{-1}B^* = B(B^*B)^{-1}B^* = P_B$.

B. Show that P_B is self-adjoint.
The adjoint of a product is the product of the adjoints in reverse order:
$P_B{}^* = \left(B(B^*B)^{-1}B^*\right)^* = B^{**}\left((B^*B)^{-1}\right)^* B^* = B\left((B^*B)^{-1}\right)^* B^*$.
The adjoint of the inverse is the inverse of the adjoint:
$B\left((B^*B)^{-1}\right)^* B^* = B\left((B^*B)^*\right)^{-1} B^*$.
Now apply the fact that the adjoint of a product is the product of the adjoints in reverse order to B^*B:
$B\left((B^*B)^*\right)^{-1} B^* = B\left((B^*B^{**})\right)^{-1} B^* = B(B^*B)^{-1}B^* = P_B$.

C. Show that P_B is a projection onto the range of B. That is, (i) if w is in the range of B, then $P_B w = w$ and (ii) if $P_B w = w$, then w is in the range of B i.e., $w = Bu$ for some u.
(i) Say w is in the range of B, i.e., $w = Bv$ for some v. Then
$P_B(w) = P_B(Bv) = \left(B(B^*B)^{-1}B^*\right)Bv = B(B^*B)^{-1}(B^*B)v = Bv = w$, so $P_B w = w$.
(ii) Conversely, if $P_B w = w$, then $w = B(B^*B)^{-1}B^*w = B\left((B^*B)^{-1}B^*w\right)$, so if $u = (B^*B)^{-1}B^*w$, then $w = Bu$.