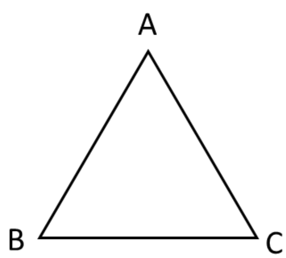
Linear Transformations and Group Representations

Homework #3 (2020-2021), Answers

Q1: Computing characters

We consider the group G of all rotations and reflections of an equilateral triangle. We will designate the group elements by the way that they permute the vertices – for example, (AB) is the reflection through vertex C and the opposite side AB that swaps vertices A and B, (ABC) is the rotation that moves A to B, B to C, and C to A. There are 6 group elements: the three pair-swaps, the two non-trivial rotations, and the identity, which we denote e.



Here we create a table of of characters several unitary representations. The first row of the table is the character of the trivial representation I, which maps every group element to 1:

representation	*g :*	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
trivial	$\chi_I(g)$:	*1*	*1*	*1*	*1*	*1*	*1*

A. Considering each element of the group as a permutation on the vertices, there is a group representation S corresponding to the sign (parity) of the permutation. What is its character?
The pair-swaps are odd parity (they are explicitly a single pair-swap); the 3-cycles are equivalent to one pair-swap followed by another (e.g., (AB) followed by (BC) is (ACB). So they have even parity.

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
trivial	$\chi_I(g)$:	1	1	1	1	1	1
sign	$\chi_S(g)$:	1	-1	-1	-1	1	1

B. Considering each element of the group as a rotation or reflection in the 2D plane, there is a group representation R corresponding to these 2×2 matrices. What is its character?

e is the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which has trace of 2. The pair-swap (BC) is a reflection across one axis (we can choose the axes however we want), so it corresponds to the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which has a trace of 0. The other pair-swaps are also flips across one axis, but after a change of coordinates – which doesn't change the trace.

The 3-cycle (ABC) corresponds to the rotation matrix $\begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}$, which has a trace of

$2\cos\frac{2\pi}{3} = -1$.

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
trivial	$\chi_I(g)$:	1	1	1	1	1	1
sign	$\chi_S(g)$:	1	-1	-1	-1	1	1
rot/ref	$\chi_R(g)$:	2	0	0	0	-1	-1

C. Considering each group element as a permutation on the vertices, each group element can be represented as a 3×3 permutation matrix. This yields a representation P. What is its character?

The group identity e corresponds to the 3×3 identity matrix, which has a trace of 3. A pair-swap corresponds to a 3×3 permutation matrix that has two symmetrical off-diagonal 1's (corresponding to the swap), and a third on-diagonal 1, such as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ for (BC). These have a trace of 1. The 3-cycles correspond to the permutation matrices $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and its inverse (or transpose), and have a trace of 0.

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
trivial	$\chi_I(g)$:	1	1	1	1	1	1
sign	$\chi_S(g)$:	1	-1	-1	-1	1	1
rot/ref	$\chi_R(g)$:	2	0	0	0	-1	-1
permutation	$\chi_P(g)$:	3	1	1	1	0	0

D. What is the character of the direct-sum representation $S \oplus R$?

$\chi_{S\oplus R} = \chi_S + \chi_R$, so:

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
trivial	$\chi_I(g)$:	1	1	1	1	1	1
sign	$\chi_S(g)$:	1	-1	-1	-1	1	1
rot/ref	$\chi_R(g)$:	2	0	0	0	-1	-1
permutation	$\chi_P(g)$:	3	1	1	1	0	0
sign $\oplus$ rotation	$\chi_{S\oplus R}(g)$:	3	-1	-1	-1	0	0

E. What is the character of the tensor-product representation $R \otimes R$?

$\chi_{R\otimes R} = {\chi_R}^2$, so:

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
trivial	$\chi_I(g)$:	1	1	1	1	1	1
sign	$\chi_S(g)$:	1	-1	-1	-1	1	1
rot/ref	$\chi_R(g)$:	2	0	0	0	-1	-1

permutation	$\chi_P(g)$:	3	1	1	1	0	0
sign $\oplus$ rot/ref	$\chi_{S\oplus R}(g)$:	3	-1	-1	-1	0	0
rot/ref $\otimes$ rot/ref	$\chi_{R\otimes R}(g)$:	4	0	0	0	1	1

F. Multiplication by a group element can be viewed as a permutation on the six members of the group. Writing these permutations as 6×6 permutation matrices yields the regular representation L. What is its character?
The character is the trace, i.e., the sum of the elements on the diagonal, which, for a permutation matrix, is the number of elements that are unchanged. Multiplication by the identity leaves all the elements unchanged, so its character is 6. Multiplication by any other element leaves no element unchanged, so every other group element has a character of 0.

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
trivial	$\chi_I(g)$:	1	1	1	1	1	1
sign	$\chi_S(g)$:	1	-1	-1	-1	1	1
rot/ref	$\chi_R(g)$:	2	0	0	0	-1	-1
permutation	$\chi_P(g)$:	3	1	1	1	0	0
sign $\oplus$ rot/ref	$\chi_{S\oplus R}(g)$:	3	-1	-1	-1	0	0
rot/ref $\otimes$ rot/ref	$\chi_{R\otimes R}(g)$:	4	0	0	0	1	1
regular	$\chi_L(g)$:	6	0	0	0	0	0

Note that the characters at the three pairwise swaps are always identical, and the characters at the two three-cycles are always identical. Why must this be?

Q2: Using characters
One result was that for any group representation A and B, the dimension of the space in which it acts trivially is given by $d(I,A)=\frac{1}{|G|}\sum_g \chi_A(g)$. For each of the above representations I, S, R, P, $S\oplus R$, $R\otimes R$, and L, compute $d(I,A)=\frac{1}{|G|}\sum_g \chi_A(g)$.

We shorten the work a bit by making use of the fact that characters are constant on the pair-swaps and on the three-cycles.

representation	g:	e	$pair-swaps(3)$	$three-cycles(2)$
trivial	$\chi_I(g)$:	1	1	1
sign	$\chi_S(g)$:	1	-1	1
rot/ref	$\chi_R(g)$:	2	0	-1
permutation	$\chi_P(g)$:	3	1	0
sign $\oplus$ rot/ref	$\chi_{S\oplus R}(g)$:	3	-1	0
rot/ref $\otimes$ rot/ref	$\chi_{R\otimes R}(g)$:	4	0	1
regular	$\chi_L(g)$:	6	0	0

$$d(I,I)=\frac{1}{6}(1+3\bullet 1+2\bullet 1)=1$$

$$d(I,S)=\frac{1}{6}(1+3\bullet(-1)+2\bullet 1)=0$$

$$d(I,R) = \frac{1}{6}(2 + 3 \bullet 0 + 2 \bullet (-1)) = 0$$

$$d(I,P) = \frac{1}{6}(3 + 3 \bullet 1 + 2 \bullet 0) = 1$$

$$d(I, S \oplus R) = \frac{1}{6}(3 + 3 \bullet (-1) + 2 \bullet 0) = 0$$

$$d(I, R \otimes R) = \frac{1}{6}(4 + 3 \bullet 0 + 2 \bullet 1) = 1$$

$$d(I,L) = \frac{1}{6}(6 + 3 \bullet 0 + 2 \bullet 0) = 1$$