Exam, 2020-2021 Questions and Solutions

Note that many of the answers are (much) more detailed than is required for full credit.

1. Basic group theory and permutations

Recall that any finite group G can be exhibited as a permutation group in a standard way: multiplication by a group element g is a maping from each group element x to a new group element gx. This is a permutation because if x and y are distinct elements of G, then gx and gy are distinct.

Recall also that any permutation can be written as a set of disjoint cycles, e..g, $(ABC)(UVWX)$ is the permutation that takes $A \to B$, $B \to C$, $B \to C$, $U \to V$, $V \to W$, $W \to X$, and $X \to U$.

Now, consider a standard representation of a group by permutations, and the permutation σ_g corresponding to a particular element g, which we assume is not the identity.

A. Show that σ_g contains no cycles of length 1.

If a group element x were in a cycle of length 1, then left-multiplication by g would map x to itself, i.e., $gx = x$. This means that g is the identity (an identity for one element is an identity for all) and this contradicts the assumption that g is not the identity.

B. Show that the cycles of σ_g all have the same length.

Say that x is in a cycle of length n. Then x, gx, g^2x, …, $g^{n-1}x$ are all distinct group elements, and $g^n x = x$, i.e., g^n is the identity but g^k is not the identify for any $1 \le k < n$, i.e., g is an element of order n. Conversely, if g is an element of order n and y is any group element, then y, gy, …, $g^{n-1}y$ are all distinct, and $g^n y = y$, so y is in a cycle of length n.

C. Show that the elements in the cycle of σ_g that contains g form a subgroup.

The elements in the cycle containing g are $\{e, g, g^2, ..., g^{n-1}\}$, which is the cyclic group of order n.

D. Say a group element has a permutation representation consisting of m cycles of length n. Determine, based on m and n, whether this permutation is an even or an odd permutation.

A permutation is even or odd based on the parity of the number of pair-swaps needed to create it. Consequently, a cycle of length n is even if (and only if) n is odd: $(a_1 a_2 a_3 ... a_n) = (a_{n-1} a_n)(a_{n-2} a_{n-1})(a_{n-3} a_{n-2})...(a_1 a_2)$ is a composition of $n-1$ permutations. A sequence of m disjoint cycles, each of length n, thus can be created from a composition of $m(n-1)$ pair-swaps. So it is even if $m(n-1)$ is even; otherwise, it is odd.

E. Show that the group elements whose permutation representations either have an even number of cycles, or cycles whose lengths are odd, form a subgroup.

According to part D, this subset is all of the group elements whose permutation representations are even. This subset contains the identity, and it inherits associativity from the parent group. It contains its inverses, as the inverse of a cycle is the same cycle in reverse order. It is closed, since a product of two even permutations is even.

2. Projections and commuting operators

Consider two projection operators, P and Q, acting in the same vector space. Further, assume that P and Q commute.

A. Under what circumstances is PQ a projection?

For A, B, and C, we use the following characterization of a projection: a self-adjoint operator X for which $X^2 = X$.

$(PQ)^* = Q^* P^* = QP = PQ$, where the first equality follows because the adjoint of a product is the product of the adjoints in reverse order; the second equality follows because P and Q, being projections, are self-adjoint; and the third equality follows because P and Q commute.

To see if PQ satisfies $(PQ)^2 = PQ$: $(PQ)^2 = PQPQ = P^2Q^2 = PQ$, where we have used commutation in the second equality and $P^2 = P$ and $Q^2 = Q$ in the third equality. So if P and Q commute, PQ is always a projection.

B. Under what circumstances is $P+Q$ a projection?

$P+Q$ is always self-adjoint: $(P+Q)^* = P^* + Q^* = P+Q$. To see if $P+Q$ satisfies $(P+Q)^2 = P+Q$:

$(P+Q)^2 = (P+Q)(P+Q) = P^2 + PQ + QP + Q^2 = P + PQ + QP + Q = P + 2PQ + Q$, so we need $PQ = 0$ -- which is not always the case. So if t P and Q commute, $P+Q$ is a projection if (and only if) $PQ = QP = 0$.

C. Under what circumstances is $P+Q-PQ$ a projection?

$P+Q-PQ$ is always self-adjoint, via parts A and B. To see if $(P+Q-PQ)^2$ satisfies $(P+Q-PQ)^2 = P+Q-PQ$:

$$
\begin{aligned}
&(P+Q-PQ)^2 = (P+Q-PQ)(P+Q-PQ) = \\
&P^2 + PQ - P^2Q + QP + Q^2 - QPQ - PQP - PQ^2 + PQPQ = \\
&P + PQ - PQ + PQ + Q - PQ - PQ - PQ + PQ = P + Q - PQ
\end{aligned}
$$

where the third equality follows because P and Q commute, and $P^2 = P$ and $Q^2 = Q$.

D. Assume that PQ is a projection (and also that they commute). Describe the range of PQ, in terms of the range of P and the range of Q (and justify).

The range of PQ is the intersection of the range of P and the range of Q.

To see that the range of PQ includes the intersection of the range of P and the range of Q: If v is in the range of P and the range of Q, then $v = Pw$ for some w, so $Pv = P^2w = Pw = v$; similarly, $Qv = v$. So $PQv = Pv = v$, and v is in the range of PQ.

To see that the intersection of the range of P and the range of Q includes the range of PQ: Since PQ is a projection, v in the range of PQ satisfies $v = PQv$. $v = P(Qv)$ displays v as being in the range of P. $v = PQv = QPv = Q(Pv)$ displays v as being in the range of Q.

E. Assume that $P+Q-PQ$ is a projection (and also that they commute). Describe the range of $P+Q-PQ$, in terms of the range of P and the range of Q (and justify).

The range of $P+Q-PQ$ is the linear span of the range of P and the range of Q.

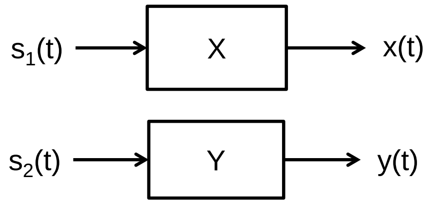
To see that the range of $P+Q-PQ$ includes the linear span of the range of P and the range of Q: If v is in the linear span of the range of P and the range of Q, then $v=x+y$, for some x in the range of P (and therefore, $x=Px$)and y in the range of Q (and therefore,). So

$$P+Q-PQ \begin{aligned} &(P+Q-PQ)(x+y)=Px+Py+Qx+Qy-PQx-PQy \\ &=x+Py+Qx+y-QPx-PQy \\ &=x+Py+Qx+y-Qx-Py \\ &=x+y \end{aligned} .$$

To see that the linear span of the range of P and the range of Q includes the range of $P+Q-PQ$: Say v is in the range of $P+Q-PQ$. Since $P+Q-PQ$ is a projection, $(P+Q-PQ)v=v$. $Pv-PQv=P(I-Q)v$ is in the range of P. Qv is in the range of Q. So $v=(P+Q-PQ)v=(P-PQ)v+Qv$ is in the linear span of the range of P and the range of Q.

3. Point processes, filters, power spectra

A. $s_1(t)$ and $s_2(t)$ are independent Poisson processes, each with rate λ, and X and Y are linear filters, with transfer functions $\tilde{X}(\omega)$ and $\tilde{Y}(\omega)$, that receive these signals as inputs. What are the power spectra of the output signals, $P_X(\omega)$ and $P_Y(\omega)$, and the cross-spectra $P_{X,Y}(\omega)$?



In general, the power spectra of the input and output of a linear system are related by multiplication by the magnitude-squared of its transfer function. The power spectrum of a Poisson process is independent of frequency, and equal to its rate. So $P_X(\omega)=\lambda\left|\tilde{X}(\omega)\right|^2$ and $P_Y(\omega)=\lambda\left|\tilde{Y}(\omega)\right|^2$.

Since $s_1(t)$ and $s_2(t)$ are independent, so are $x(t)$ and $y(t)$. So the cross-spectrum is zero.

B. For any two random signals $a(t)$ and $b(t)$, we can consider the sum signal $(a+b)$ defined by $(a+b)(t)=a(t)+b(t)$ and the difference signal $(a-b)$ defined by $(a-b)(t)=a(t)-b(t)$. Show that the real part of the cross-spectrum of $a(t)$ and $b(t)$ is given by $\mathrm{Re}\left\{P_{A,B}(\omega)\right\}=\frac{1}{4}\left(P_{A+B}(\omega)-P_{A-B}(\omega)\right)$.

The power spectrum is a limit of magnitude-squareds of spectral estimates $P_Z(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|F(z,\omega,T,T_0)\right|^2\right\rangle$, where the spectral estimates are defined by $F(z,\omega,T,T_0)=\int_{T_0}^{T_0+T} z(t)e^{-i\omega t}dt$. So

$$P_{A+B}(\omega)-P_{A-B}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|F(a+b,\omega,T,T_0)\right|^2-\left|F(a-b,\omega,T,T_0)\right|^2\right\rangle.$$

Spectral estimates are linear in the signals:

$$F(a+ub,\omega,T,T_0)=\int_{T_0}^{T_0+T}\left(a(t)+ub(t)\right)e^{-i\omega t}dt=F(a,\omega,T,T_0)+uF(b,\omega,T,T_0).$$

So

$$\begin{aligned}
&\left|F(a+ub,\omega,T,T_0)\right|^2=F(a+ub,\omega,T,T_0)\overline{F(a+ub,\omega,T,T_0)}\\
&=\left(F(a,\omega,T,T_0)+uF(b,\omega,T,T_0)\right)\overline{\left(F(a,\omega,T,T_0)+uF(b,\omega,T,T_0)\right)}\\
&=\left|F(a,\omega,T,T_0)\right|^2+\left|u\right|^2\left|F(b,\omega,T,T_0)\right|^2+\bar{u}F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}+u\overline{F(a,\omega,T,T_0)}F(b,\omega,T,T_0)
\end{aligned}.$$

(We will use this result in Part C).

Choosing $u=1$:

$$\left|F(a+b,\omega,T,T_0)\right|^2=\left|F(a,\omega,T,T_0)\right|^2+\left|F(b,\omega,T,T_0)\right|^2+F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}+\overline{F(a,\omega,T,T_0)}F(b,\omega,T,T_0).$$

Choosing $u=-1$:

$$\left|F(a-b,\omega,T,T_0)\right|^2=\left|F(a,\omega,T,T_0)\right|^2+\left|F(b,\omega,T,T_0)\right|^2-F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}-\overline{F(a,\omega,T,T_0)}F(b,\omega,T,T_0).$$

Combining the above:

$$\left|F(a+b,\omega,T,T_0)\right|^2-\left|F(a-b,\omega,T,T_0)\right|^2=2\left(F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}+\overline{F(a,\omega,T,T_0)}F(b,\omega,T,T_0)\right).$$

Since for any complex z, $z+\overline{z}=2\operatorname{Re}z$,

$$\left|F(a+b,\omega,T,T_0)\right|^2-\left|F(a-b,\omega,T,T_0)\right|^2=4\operatorname{Re}\left\{F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}\right\}.$$

This is four times the real part of the spectral estimate required for the cross-spectrum, $P_{A,B}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}\right\rangle$, so

$$\operatorname{Re}\left\{P_{A,B}(\omega)\right\}=\lim_{T\to\infty}\frac{1}{T}\left\langle\operatorname{Re}\left\{F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}\right\}\right\rangle$$
$$=\lim_{T\to\infty}\frac{1}{T}\left\langle\frac{1}{4}\left(\left|F(a+b,\omega,T,T_0)\right|^2-\left|F(a-b,\omega,T,T_0)\right|^2\right)\right\rangle=\frac{1}{4}\left(P_{A+B}(\omega)-P_{A-B}(\omega)\right).$$

C. For any two random signals $a(t)$ and $b(t)$, we can also consider the signals $(a+ib)$ defined by $(a+ib)(t)=a(t)+ib(t)$ and $(a-ib)$ defined by $(a-ib)(t)=a(t)-ib(t)$. They have complex values, but still, their power spectra can be defined as limits of the magnitude-squared of their spectral estimates. Show that the imaginary part of the cross-spectrum of $a(t)$ and $b(t)$ is given by $\operatorname{Im}\left\{P_{A,B}(\omega)\right\}=\frac{1}{4}\left(P_{A+iB}(\omega)-P_{A-iB}(\omega)\right)$.

We use the framework of part B. Choosing $u=i$:

$$\left|F(a+ib,\omega,T,T_0)\right|^2=\left|F(a,\omega,T,T_0)\right|^2+\left|F(b,\omega,T,T_0)\right|^2-iF(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}+i\overline{F(a,\omega,T,T_0)}F(b,\omega,T,T_0)$$

Choosing $u=-i$

$$\left|F(a-ib,\omega,T,T_0)\right|^2=\left|F(a,\omega,T,T_0)\right|^2+\left|F(b,\omega,T,T_0)\right|^2+iF(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}-i\overline{F(a,\omega,T,T_0)}F(b,\omega,T,T_0)$$

Combining the above:

$$\left|F(a+ib,\omega,T,T_0)\right|^2-\left|F(a-ib,\omega,T,T_0)\right|^2=2i\left(-F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}+\overline{F(a,\omega,T,T_0)}F(b,\omega,T,T_0)\right).$$

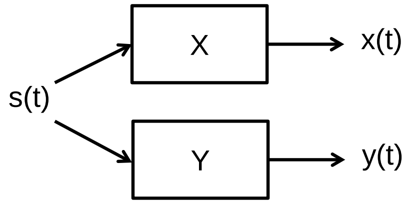
Since for any complex z, $-z+\overline{z}=-2i\operatorname{Im}z$,

$$\left|F(a+ib,\omega,T,T_0)\right|^2-\left|F(a-ib,\omega,T,T_0)\right|^2=4\operatorname{Im}\left\{F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}\right\}.$$

This is four times the imaginary part of the spectral estimate required for the cross-spectrum, $P_{A,B}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}\right\rangle$, so

$$\operatorname{Im}\left\{P_{A,B}(\omega)\right\}=\lim_{T\to\infty}\frac{1}{T}\left\langle\operatorname{Im}\left\{F(a,\omega,T,T_0)\overline{F(b,\omega,T,T_0)}\right\}\right\rangle$$
$$=\lim_{T\to\infty}\frac{1}{T}\left\langle\frac{1}{4}\left(\left|F(a+ib,\omega,T,T_0)\right|^2-\left|F(a-ib,\omega,T,T_0)\right|^2\right)\right\rangle=\frac{1}{4}\left(P_{A+iB}(\omega)-P_{A-iB}(\omega)\right).$$

D. Use the results of B and C (even if you didn't demonstrate them) to determine the cross-spectrum of X and Y for the following system. Here, the linear filters share a common input $s(t)$, a Poisson process of rate λ.



P_{X+uY} is the power spectrum of the signal $x(t)+uy(t)$, i.e., the power spectrum of a a linear system with transfer function $\tilde{X}(\omega)+u\tilde{Y}(\omega)$, driven by a Poisson process. Therefore,

$$P_{X+uY}(\omega)=\lambda\left|\tilde{X}(\omega)+u\tilde{Y}(\omega)\right|^2=\lambda\left(\tilde{X}(\omega)+u\tilde{Y}(\omega)\right)\overline{\left(\tilde{X}(\omega)+u\tilde{Y}(\omega)\right)}$$

$$=\lambda\left(\tilde{X}(\omega)\overline{\tilde{X}(\omega)}+\bar{u}\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+u\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)+u\bar{u}\tilde{Y}(\omega)\overline{\tilde{Y}(\omega)}\right)$$

$$=\lambda\left(\left|\tilde{X}(\omega)\right|^2+|u|^2\left|\tilde{Y}(\omega)\right|^2+\bar{u}\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+u\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)\right).$$

$$=P_X(\omega)+|u|^2P_Y(\omega)+\lambda\left(\bar{u}\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+u\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)\right)$$

Taking $u=\pm 1$ and using the result of part B:

$$P_{X+Y}(\omega)=P_X(\omega)+P_Y(\omega)+\lambda\left(\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)\right),$$

$$P_{X-Y}(\omega)=P_X(\omega)+P_Y(\omega)+\lambda\left(\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}-\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)\right).$$

$$\mathrm{Re}\left\{P_{X,Y}(\omega)\right\}=\frac{1}{4}\left(P_{X+Y}(\omega)-P_{X-Y}(\omega)\right)=\frac{\lambda}{2}\left(\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)\right)=\lambda\,\mathrm{Re}\left\{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}\right\}.$$

Taking $u=\pm i$ and using the result of part C:

$$P_{X+iY}(\omega)=P_X(\omega)+P_Y(\omega)+\lambda\left(-i\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+i\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)\right),$$

$$P_{X-iY}(\omega)=P_X(\omega)+P_Y(\omega)+\lambda\left(i\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}-i\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)\right).$$

$$\mathrm{Im}\left\{P_{X,Y}(\omega)\right\}=\frac{1}{4}\left(P_{X+iY}(\omega)-P_{X+iY}(\omega)\right)=\frac{\lambda}{2}\left(-i\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+i\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)\right)=\lambda\,\mathrm{Im}\left\{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}\right\}.$$

So, since $\mathrm{Re}\left\{P_{X,Y}(\omega)\right\}=\lambda\,\mathrm{Re}\left\{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}\right\}$ and $\mathrm{Im}\left\{P_{X,Y}(\omega)\right\}=\lambda\,\mathrm{Im}\left\{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}\right\}$, it follows that $P_{X,Y}(\omega)=\lambda\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}$.

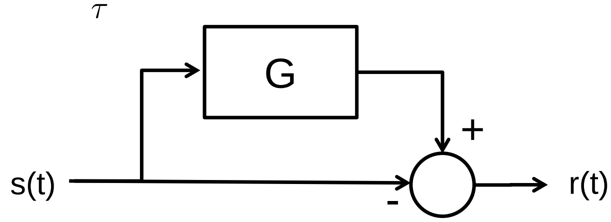
E. Same as part D, but now, $s(t)$ is an arbitrary signal, whose power spectrum is $P_S(\omega)$.

Everything proceeds as in Parts A and D, with $P_S(\omega)$ replacing λ. In particular, $P_X(\omega)=P_S(\omega)\left|\tilde{X}(\omega)\right|^2$, $P_Y(\omega)=P_S(\omega)\left|\tilde{Y}(\omega)\right|^2$, and

$$P_{X+uY}(\omega)=P_S(\omega)\left|\tilde{X}(\omega)+u\tilde{Y}(\omega)\right|^2=P_X(\omega)+|u|^2P_Y(\omega)+P_S(\omega)\left(\bar{u}\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+u\overline{\tilde{X}(\omega)}\tilde{Y}(\omega)\right).$$

Just as in D, this leads to $\mathrm{Re}\left\{P_{X,Y}(\omega)\right\}=P_S(\omega)\,\mathrm{Re}\left\{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}\right\}$, $\mathrm{Im}\left\{P_{X,Y}(\omega)\right\}=P_S(\omega)\,\mathrm{Im}\left\{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}\right\}$, and $P_{X,Y}(\omega)=P_S(\omega)\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}$.

F. What is the transfer function of the following system, where G is a linear filter with impulse response $G(t)=\frac{2}{\tau}e^{-t/\tau}$?



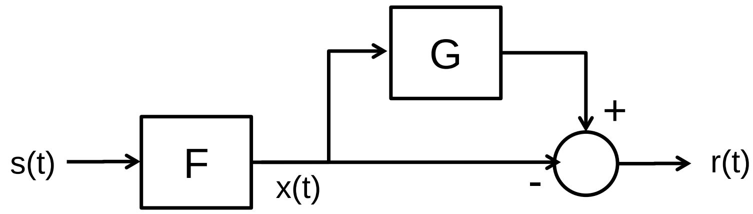
First, $\tilde{G}(\omega)=\int_0^\infty e^{-i\omega t}G(t)dt=\frac{2}{\tau}\int_0^\infty e^{-i\omega t}e^{-t/\tau}dt=\frac{2}{\tau}\int_0^\infty e^{-(i\omega+1/\tau)t}dt=\frac{2}{\tau}\left(-\frac{1}{i\omega+1/\tau}\right)e^{-(i\omega+1/\tau)t}\Big|_0^\infty$.

$$=\frac{2}{\tau}\left(\frac{1}{i\omega+1/\tau}\right)=\frac{2}{1+i\omega\tau}$$

The transfer function of the composite system, $\tilde{L}(\omega)$, is given by

$$\tilde{L}(\omega)=\tilde{G}(\omega)-1=\frac{2}{1+i\omega\tau}-1=\frac{2-(1+i\omega\tau)}{1+i\omega\tau}=\frac{1-i\omega\tau}{1+i\omega\tau}.$$

G. Now consider the following system, where $s(t)$ has power spectrum $P_S(\omega)$, F is a linear filter with transfer function $\tilde{F}(\omega)$, and G is as above. What are the power spectra of $x(t)$ and $r(t)$?



$P_X(\omega)=\left|\tilde{F}(\omega)\right|^2 P_S(\omega)$.

$P_R(\omega)=\left|\tilde{L}(\omega)\right|^2 P_X(\omega)$, where (from part F) $\left|\tilde{L}(\omega)\right|^2=\tilde{L}(\omega)\overline{\tilde{L}(\omega)}=\frac{1-i\omega\tau}{1+i\omega\tau}\frac{1+i\omega\tau}{1-i\omega\tau}=1$. So

$P_R(\omega)=P_X(\omega)=\left|\tilde{F}(\omega)\right|^2 P_S(\omega)$ as well.

That is, a nontrivial linear filter can leave the power spectrum unchanged. Equivalently, two distinguishable linear filters can yield the indistinguishable power spectra.

4. Principal components

Consider principal components analysis of a dataset consisting of k *time series* $\vec{y}_j$*, each of length* n *(*$n \gg k$*, assembled into an* $n \times k$ *matrix* Y*. What predictable effects will the following manipulations have on the number and size of principal components? Justify your answer. If there is a predictable effect on the size of the principal components, indicate that as well.*

A. Reversing the time points

The number of principal components is determined by the nonzero eigenvalues of Y^*Y (or YY^*), and their contributions are given by the size of the eigenvalues. So reversing the time points has no effect. This replaces Y by $Z = PY$, where P is a permutation matrix that looks like this:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$ Since $P^*P = I$, $Z^*Z = (PY)^*(PY) = Y^*P^*PY = Y^*Y$.

B. Adjoining a new time series equal to the average of the $\vec{y}_j$*.*

This will not change the number of principal components, as the average of the time series is always contained in their linear span. The eigenvalues will not change in a predictable way.

C. Subtracting the average time series from each of the $\vec{y}_j$*.*

Subtracting the mean time series creates a linear dependence of the time series, so the dimension of the linear span will typically be reduced by 1. (If the mean was already zero, then there will be no effect.) More formally, subtracting the mean time series replaces Y by $Z = YM$, where $M = I - \frac{1}{k}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$. So

$Z^*Z = (YM)^*(YM) = MY^*YM$. M has an eigenvector of eigenvalue zero (a column of 1's), so therefore so does Z^*Z. Overall, the eigenvalues will be smaller, since some of the variance has now been removed. But the details of how the eigenvalues change cannot be predicted.

D. Replacing the $\vec{y}_j$ *by their pairwise sums and differences (assuming* k *is even).*

Replacing the time series by paired means and differences corresponds to replacing Y by $Z = YA$, where A is a block-diagonal matrix $A = \begin{pmatrix} +1 & +1 & 0 & 0 & \cdots & 0 & 0 \\ +1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & +1 & +1 & & 0 & 0 \\ 0 & 0 & +1 & -1 & & 0 & 0 \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & +1 & +1 \\ 0 & 0 & 0 & 0 & \cdots & +1 & -1 \end{pmatrix}$. Note that $A^*A = 2I$. So

$Z^*Z = \left(AY\right)^* (AY) = Y^* A^* AY = 2Y^*Y$. The number of principal components will not change, and all of the eigenvectors will double – so the relative sizes of the contributions of the principal components will not change.

E. Adjoining a new time series equal to the point-by-point square of the first time series

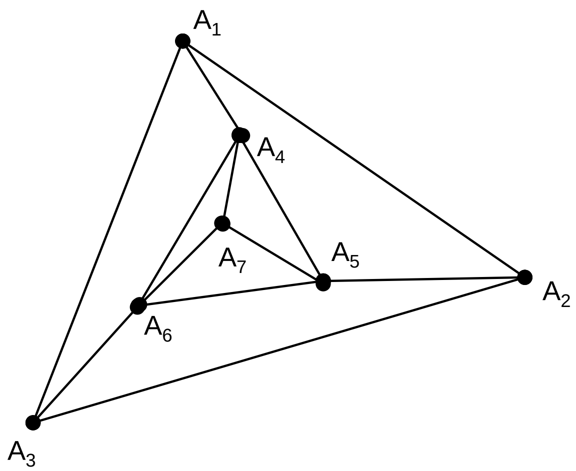
Adjoining a new time series equal to the point-by-point square of the first time series will typically increase the number of principal components by 1, since the new time series need not be in the linear span of the existing ones. The eigenvalues will not change in a predictable way.

F. Subtracting the mean and linear trend from each of the $\vec{y}_j$.

Subtracting the mean and linear trend is a linear transformation – in fact, it is a projection, since applying this transformation twice is the same as applying it once. If either the constant vector or the ramp are in the space spanned by the $\vec{y}_j$, then the dimensionality of the span of the $\vec{y}_j$ will be reduced by 1; if both are in the span, the dimensionality will be reduced by 2. The eigenvalues will not change in a predictable way.

5. Graph Laplacian

Consider the following graph.



A. What is its graph Laplacian, L*.*

From the definition $L = D - A$, the difference of the diagonal matrix of the degrees (D) and the adjacency matrix,

$$L = \begin{pmatrix} +3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & +3 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & +3 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & +4 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & +4 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & +4 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & +3 \end{pmatrix}$$

B. Based on the symmetry of the graph, write a permutation matrix $P \neq I$ *that commutes with* L*, for which* $P^3 = I$*.*

Since the connectivity of the graph is unchanged following the permutation $(A_1 A_2 A_3)(A_4 A_5 A_6)$, we can take

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

C. Determine the eigenvalues and eigenvectors of P*.*

Since $P^3 = I$, an eigenvector v with eigenvalue λ must satisfy $v = Iv = P^3 v = \lambda^3 v$, so $\lambda^3 = 1$, and $\lambda = 1$ or $\lambda = \omega$ or $\lambda = \omega^2$, where $\omega = e^{2\pi i/3}$.

Regarding eigenvectors: P is block-diagonal (two 3×3 blocks and one 1×1 block), so it acts independently on two three-dimensional subspaces and a one-dimensional subspace. If, for example, $Pv=\lambda v$ -- in coordinates,

$$P\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}=\lambda\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

-- , then $x_2=\lambda x_1$, $x_3=\lambda x_2$, $x_1=\lambda x_3$, $x_5=\lambda x_4$, $x_6=\lambda x_5$, $x_4=\lambda x_6$, and $x_7=\lambda x_7$, so

$$v=\begin{pmatrix} a \\ \lambda a \\ \lambda^2 a \\ b \\ \lambda b \\ \lambda^2 b \\ c \end{pmatrix}$$

, where $\lambda\in\{1,\omega,\omega^2\}$, and at least one of a, b, or c are nonzero. So $\lambda^3=1$. From $x_7=\lambda x_7$, if $\lambda\neq 1$, then $c=0$.

Choosing $(a,b,c)=(1,0,0)$ or $(a,b,c)=(0,1,0)$ for each $\lambda\in\{1,\omega,\omega^2\}$, or $(a,b,c)=(0,0,1)$ for $\lambda=1$ thus displays seven eigenvectors of P, as consisting of two pairs (nonzero a or nonzero b for each $\lambda\in\{\omega,\omega^2\}$), and one triplet (nonzero a, b. or c for $\lambda=1$). Linear combinations of eigenvectors with a shared eigenvalue are also eigenvectors. That is, the vector space in which P and L act is decomposed into two two-dimensional subspaces, one for each $\lambda\in\{\omega,\omega^2\}$, and a three-dimensional subspace for $\lambda=1$.

D. Using $PL=LP$, determine the eigenvalues of L.

The eigenvectors for L must lie wholly within each of the three subspaces outlined above. (Other than an intuitive argument based on symmetry), the formal reason for this is the following. If $Pv=\lambda v$, then $P(Lv)=L(Pv)=\lambda(Lv)$. So if v is in the eigenspace corresponding to λ, then so is Lv. So we can see how L acts on the eigenspace corresponding to λ (where $\lambda\in\{1,\omega,\omega^2\}$) by choosing the above eigenvectors as a basis:

$$u_1(\lambda)=\begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ u_2(\lambda)=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \lambda \\ \lambda^2 \\ 0 \end{pmatrix},\text{ and } u_3=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We then calculate how L maps a generic element of the λ-eigenspace, $au_1+bu_2+cu_3$ (with $c=0$ unless $\lambda=1$):

$$L(au_1+bu_2+cu_3)=\begin{pmatrix}+3&-1&-1&-1&0&0&0\\-1&+3&-1&0&-1&0&0\\-1&-1&+3&0&0&-1&0\\-1&0&0&+4&-1&-1&-1\\0&-1&0&-1&+4&-1&-1\\0&0&-1&-1&-1&+4&-1\\0&0&0&-1&-1&-1&+3\end{pmatrix}\begin{pmatrix}a\\\lambda a\\\lambda^2 a\\b\\\lambda b\\\lambda^2 b\\c\end{pmatrix}=\begin{pmatrix}(3-\lambda-\lambda^2)a-b\\(-1+3\lambda-\lambda^2)a-\lambda b\\(-1-\lambda-3\lambda^2)a-\lambda^2 b\\-a+(4-\lambda-\lambda^2)b-c\\-a\lambda+(-1+4\lambda-\lambda^2)b-c\\-a^2\lambda+(-1-\lambda+4\lambda^2)b-c\\(-1-\lambda-\lambda^2)b+3c\end{pmatrix}$$

$$=\left((3-\lambda-\lambda^2)a-b\right)u_1+\left(-a+(4-\lambda-\lambda^2)b-c\right)u_2+\left((-1-\lambda-\lambda^2)b+3c\right)u_3$$

Note that $\lambda^3=1$ was crucial for this, and this holds for $\lambda\in\{1,\omega,\omega^2\}$.

For $(a,b,c)=(1,0,0)$, $Lu_1=(3-\lambda-\lambda^2)u_1-u_2$; for $(a,b,c)=(0,1,0)$, $Lu_2=-u_1+(4-\lambda-\lambda^2)u_2+(-1-\lambda-\lambda^2)u_3$.

So, in the eigenspaces of P corresponding to $\lambda\in\{\omega,\omega^2\}$, L acts like the transformation

$$L_\lambda=\begin{pmatrix}3-\lambda-\lambda^2&-1\\-1&4-\lambda-\lambda^2\end{pmatrix}=\begin{pmatrix}+4&-1\\-1&+5\end{pmatrix}, \text{ since for these } \lambda,\ 1+\lambda+\lambda^2=0.$$

For $\lambda=1$ we also have to consider the third basis element, $(a,b,c)=(0,0,1)$. $L_1u_3=-u_2+3u_3$. So, in the 1-eigenspace of P, L acts like the transformation

$$L_1=\begin{pmatrix}3-\lambda-\lambda^2&-1&0\\-1&4-\lambda-\lambda^2&-1\\0&-3&+3\end{pmatrix}=\begin{pmatrix}+1&-1&0\\-1&+2&-1\\0&-3&+3\end{pmatrix}.$$

We therefore need to find the eigenvalues of the above three transformations.

We know that one of the eigenvalues must be zero, since the graph is connected. Since L_λ is not singular (its determinant is $4\cdot5-(-1)\cdot(-1)=19$, it follows that this must be an eigenvalue of L_1. So the determinant of L_1 must be zero. This is a helpfiul check.

This simplifies writing its characteristic equation:

$$\det(\mu I-L_1)=\det\begin{pmatrix}\mu-1&1&0\\1&\mu-2&1\\0&3&\mu-3\end{pmatrix}=(\mu-1)(\mu-2)(\mu-3)-(\mu-1)\cdot1\cdot3-1\cdot1\cdot(\mu-3)$$

$$=\left(\mu^3-6\mu^2+11\mu-6\right)-(3\mu-3)-(\mu-3)=\mu^3-6\mu^2+7\mu=\mu(\mu^2-6\mu+7)$$

. So, in addition to the guaranteed eigenvalue of 0, the other two eigenvalues are the roots of the quadratic $\mu^2-6\mu+7=0$, namely, $\mu=\dfrac{6\pm\sqrt{36-28}}{2}=3\pm\sqrt{2}$.

The other four eigenvalues of L are the two eigenvalues of L_λ, each occurring twice – once for $\lambda=\omega$ and once for $\lambda=\omega^2=\overline{\omega}$. The characteristic equation for L_λ is $\det(\mu I-L_\lambda)=\det\begin{pmatrix}\mu-4 & 1\\ 1 & \mu-5\end{pmatrix}=(\mu-4)(\mu-5)-1\cdot 1=\mu^2-9\mu+19$. Its roots, via the quadratic formula, are $\mu=\dfrac{9\pm\sqrt{81-76}}{2}=\dfrac{9\pm\sqrt{5}}{2}$.

So the eigenvalues of L, in ascending order, are $\left\{0,\ 3-\sqrt{2},\ \dfrac{9-\sqrt{5}}{2}\ (twice)\ ,3+\sqrt{2},\ \dfrac{9+\sqrt{5}}{2}\ (twice)\right\}$.

E. Find the eigenvectors of L.

To find the eigenvectors, we use the above decomposition into a pair of 2-dimensional subspaces and a three-dimensional subspace. That is, we find the eigenvectors of L_λ and L_1 above, and then reconstitute them in the full 7-dimensional space from the definitions of u_1, u_2, and u_3.

In L_1, an eigenvector is a solution of $(\mu I-L_1)\begin{pmatrix}a\\ b\\ c\end{pmatrix}=0$, i.e., $\begin{pmatrix}\mu-1 & 1 & 0\\ 1 & \mu-2 & 1\\ 0 & 3 & \mu-3\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=0$, for $\mu=\{0,3\pm\sqrt{2}\}$.

$\mu=0$ corresponds to the guaranteed uniform eigenvector, $(a,b,c)=(1,1,1)$, which (using $au_1+bu_2+cu_3$ to reconstitute the 7-vector) is $\begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1\end{pmatrix}^T$.

For the eigenvectors corresponding to nonzero μ, there are no obvious shortcuts and we need to solve $\begin{pmatrix}\mu-1 & 1 & 0\\ 1 & \mu-2 & 1\\ 0 & 3 & \mu-3\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=0$. It suffices to solve the equations corresponding to the first two rows, as the third equation is linearly dependent on the first two (since the determinant is zero). So we solve $\begin{cases}(\mu-1)a+b=0\\ a+(\mu-2)b+c=0\end{cases}$. The first equation yields $b=(1-\mu)a$; the second yields $c=-a-(\mu-2)b=-a+(\mu-2)(\mu-1)a=(\mu^2-3\mu+1)a$. For $\mu=3\pm\sqrt{2}$, this is $(a,b,c)=\left(1,-2\mp\sqrt{2},3\pm 3\sqrt{2}\right)a$. Reconstituting the eigenvectors with $au_1+bu_2+cu_3$ yields the two eigenvectors $\begin{pmatrix}1 & 1 & 1 & -2\mp\sqrt{2} & -2\mp\sqrt{2} & -2\mp\sqrt{2} & 3+3\sqrt{2}\end{pmatrix}^T a$, corresponding to $\mu=3\pm\sqrt{2}$.

In L_λ, we need to solve $(\mu I-L_\lambda)\begin{pmatrix}a\\ b\end{pmatrix}=0$, i.e., $\begin{pmatrix}\mu-4 & 1\\ 1 & \mu-5\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=0$, for $\mu=\dfrac{9\pm\sqrt{5}}{2}$. We only need to solve the equation corresponding to the first row, as the two equations are linearly dependant (since the determinant is zero). That is, $(\mu-4)a+b=0$, i.e., the eigenvectors are given by $(a,b)=\left(1,4-\mu\right)a=\left(1,\dfrac{-1\mp\sqrt{5}}{2}\right)a$. Using $au_1+bu_2+cu_3$, this yields eigenvectors $\begin{pmatrix}1 & \omega & \omega^2 & \dfrac{9\pm\sqrt{5}}{2} & \dfrac{9\pm\sqrt{5}}{2}\omega & \dfrac{9\pm\sqrt{5}}{2}\omega^2 & 0\end{pmatrix}^T a$.