Exam, 2020-2021 Questions

Each question has multiple small subparts, for a total of 27 subparts.

Each is worth 2 points, so partial credit is possible.

Do at least 30 points (15 subparts), more if you want.

A guide to the dependencies:
1: A, B, C, D: no dependencies but best to do in order. E depends on D.
2. A, B, C, D, E: no dependencies.
3. A, B, C, D, E: no dependencies but best to do in order. G depends on F; these are independent A-E.
4. A, B, C, D, E, F: no dependencies.
5. A, B, C, D, E: serial dependence.

1. Basic group theory and permutations

Recall that any finite group G can be exhibited as a permutation group in a standard way: multiplication by a group element g is a maping from each group element x to a new group element gx. This is a permutation because if x and y are distinct elements of G, then gx and gy are distinct.

Recall also that any permutation can be written as a set of disjoint cycles, e.g, $(ABC)(UVWX)$ is the permutation that takes $A \to B$, $B \to C$, $B \to C$, $U \to V$, $V \to W$, $W \to X$, and $X \to U$.

Now, consider a standard representation of a group by permutations, and the permutation σ_g corresponding to a particular element g, which we assume is not the identity.

A. Show that σ_g contains no cycles of length 1.

B. Show that the cycles of σ_g all have the same length.

C. Show that the elements in the cycle of σ_g t hat contains g form a subgroup.

D. Say a group element has a permutation representation consisting of m cycles of length n. Determine, based on m and n, whether this permutation is an even or an odd permutation.

E. Show that the group elements whose permutation representations either have an even number of cycles, or cycles whose lengths are odd, form a subgroup.

2. Projections and commuting operators

Consider two projection operators, P and Q, acting in the same vector space. Further, assume that P and Q commute.

A. Under what circumstances is PQ a projection?

B. Under what circumstances is $P+Q$ a projection?

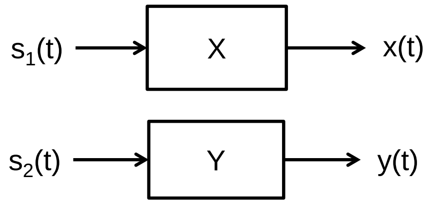
C. Under what circumstances is $P+Q-PQ$ a projection?

D. Assume that PQ is a projection (and also that they commute). Describe the range of PQ, in terms of the range of P and the range of Q (and justify).

E. Assume that $P+Q-PQ$ is a projection (and also that they commute). Describe the range of $P+Q-PQ$, in terms of the range of P and the range of Q (and justify).

3. Point processes, filters, power spectra

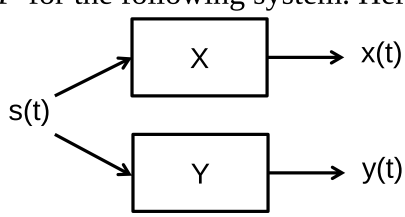
A. $s_1(t)$ and $s_2(t)$ are independent Poisson processes, each with rate λ, and X and Y are linear filters, with transfer functions $\tilde{X}(\omega)$ and $\tilde{Y}(\omega)$, that receive these signals as inputs. What are the power spectra of the output signals, $P_X(\omega)$ and $P_Y(\omega)$, and the cross-spectra $P_{X,Y}(\omega)$?



B. For any two random signals $a(t)$ and $b(t)$, we can consider the sum signal $(a+b)$ defined by $(a+b)(t)=a(t)+b(t)$ and the difference signal $(a-b)$ defined by $(a-b)(t)=a(t)-b(t)$. Show that the real part of the cross-spectrum of $a(t)$ and $b(t)$ is given by $\mathrm{Re}\left\{P_{A,B}(\omega)\right\}=\frac{1}{4}\left(P_{A+B}(\omega)-P_{A-B}(\omega)\right)$.

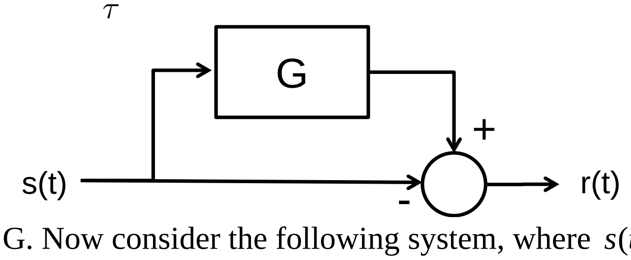
C. For any two random signals $a(t)$ and $b(t)$, we can also consider the signals $(a+ib)$ defined by $(a+ib)(t)=a(t)+ib(t)$ and $(a-ib)$ defined by $(a-ib)(t)=a(t)-ib(t)$. They have complex values, but still, their power spectra can be defined as limits of the magnitude-squared of their spectral estimates. Show that the imaginary part of the cross-spectrum of $a(t)$ and $b(t)$ is given by $\mathrm{Im}\left\{P_{A,B}(\omega)\right\}=\frac{1}{4}\left(P_{A+iB}(\omega)-P_{A-iB}(\omega)\right)$.

D. Use the results of B and C (even if you didn't demonstrate them) to determine the cross-spectrum of X and Y for the following system. Here, the linear filters share a common input $s(t)$, a Poisson process of rate λ.

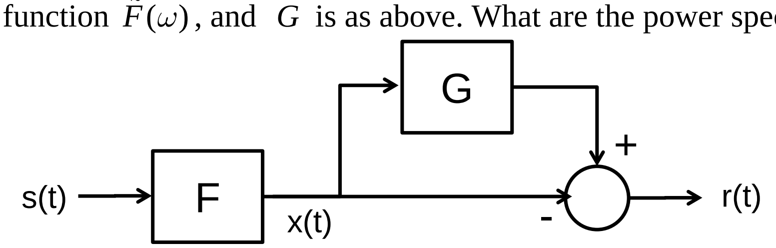


E. Same as part D, but now, $s(t)$ is an arbitrary signal, whose power spectrum is $P_S(\omega)$.

F. What is the transfer function of the following system, where G is a linear filter with impulse response $G(t)=\frac{2}{\tau}e^{-t/\tau}$?



G. Now consider the following system, where $s(t)$ has power spectrum $P_S(\omega)$, F is a linear filter with transfer function $\tilde{F}(\omega)$, and G is as above. What are the power spectra of $x(t)$ and $r(t)$?



4. Principal components

Consider principal components analysis of a dataset consisting of k time series $\vec{y}_j$, each of length n ($n \gg k$, assembled into an $n \times k$ matrix Y. What predictable effects will the following manipulations have on the number and size of principal components? Justify your answer. If there is a predictable effect on the size of the principal components, indicate that as well.

A. Reversing the time points

B. Adjoining a new time series equal to the average of the $\vec{y}_j$.

C. Subtracting the average time series from each of the $\vec{y}_j$.

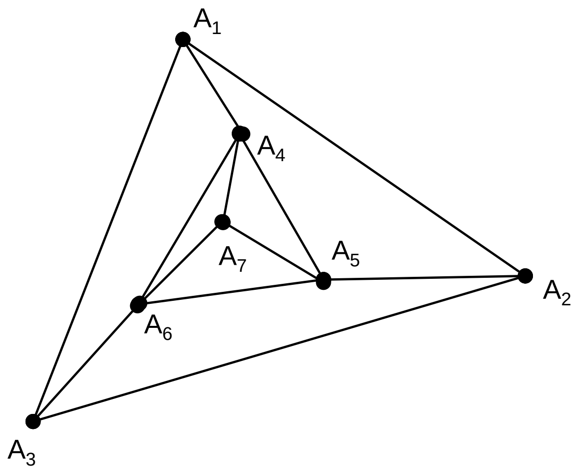
D. Replacing the $\vec{y}_j$ by their pairwise sums and differences (assuming k is even).

E. Adjoining a new time series equal to the point-by-point square of the first time series

F. Subtracting the mean and linear trend from each of the $\vec{y}_j$.

5. Graph Laplacian

Consider the following graph.



A. What is its graph Laplacian, L.

B. Based on the symmetry of the graph, write a permutation matrix $P \neq I$ that commutes with L, for which $P^3 = I$.

C. Determine the eigenvalues and eigenvectors of P.

D. Using $PL = LP$, determine the eigenvalues of L.

E. Find the eigenvectors of L.