Groups, Fields, and Vector Spaces

Homework #2 (2022-2023), Answers

Q1: The Group Algebra

Here we define a "group algebra" and see some of its basic properties.

A group algebra is the free vector space of functions on a group, along with an additional operation on the vectors that relies on the group operation. Specifically, let $S = \{e, \sigma, \tau, ...\}$ *be a group, and* f *and* g *are functions from* S *to a field* k*. The vector space operations are defined as before: addition of vectors,* $(f+g)(s) = f(s) + g(s)$*, where the addition is in* k*, and* αf*; and scalar multiplication of vectors* $(\alpha f)(s) = \alpha \cdot (f(s))$ *where the multiplication on the right is in the field* k*.*

*To make this into an algebra, we define the new operation for composing vector space elements, here denoted *. We define this on the "one-hot" basis for the free vector space and then extend by linearity to the whole space. Say* f_σ *is an element of the one-hot basis, i.e., the function on the group for which* $f_\sigma(\tau) = 1$ *for* $\tau = \sigma$ *and 0 for* $\tau \neq \sigma$*. Then* $f_\sigma * g$ *is the function on the group for which* $(f_\sigma * g)(\tau) = g(\sigma^{-1}\tau)$*. (Q2, Q3, and Q4 show why we chose this definition, rather than multiplying by* σ *on the right, or not using the inverse.)*

A. Show, for any element g *of the group algebra, and one-hot basis elements* f_{σ_1} *and* f_{σ_2}*, that* $f_{\sigma_1} * (f_{\sigma_2} * g) = f_{\sigma_1\sigma_2} * g$*.*

Evaluate at a typical element τ:

$$\left(f_{\sigma_1} * \left(f_{\sigma_2} * g\right)\right)(\tau) = \left(f_{\sigma_2} * g\right)(\sigma_1^{-1}\tau) = g\left(\sigma_2^{-1}(\sigma_1^{-1}\tau)\right) = g\left((\sigma_2^{-1}\sigma_1^{-1})\tau\right) = g\left((\sigma_1\sigma_2)^{-1}\tau\right) = \left(f_{\sigma_1\sigma_2} * g\right)(\tau)$$

B. Show, for one-hot basis elements f_{σ_1} *and* f_{σ_2}*, that* $f_{\sigma_1} * f_{\sigma_2} = f_{\sigma_1\sigma_2}$*.*

Work out the value of $h = f_{\sigma_1} * f_{\sigma_2}$ at a typical group element τ. $h(\tau) = f_{\sigma_2}(\sigma_1^{-1}\tau)$: This is zero unless $\sigma_1^{-1}\tau = \sigma_2$, and 1 when equality holds. But $\sigma_1^{-1}\tau = \sigma_2$ is equivalent to $\tau = \sigma_1\sigma_2$, so $f_{\sigma_1} * f_{\sigma_2} = f_{\sigma_1\sigma_2}$.

*C. Show that * is associative.*

Combining A and B shows that $f_{\sigma_1} * (f_{\sigma_2} * g) = f_{\sigma_1\sigma_2} * g = (f_{\sigma_1} * f_{\sigma_2}) * g$. Since any member of the group algebra can be written as $h = \sum_{\sigma \in S} h(\sigma) f_\sigma$, this extends, via linearity, to the group algebra.

*D. Is * commutative?*

No, if the underlying group is not commutative – since $f_{\sigma_1} * f_{\sigma_2} = f_{\sigma_1\sigma_2}$ but $f_{\sigma_2} * f_{\sigma_1} = f_{\sigma_2\sigma_1}$.

E. What is the identity element for addition in the group algebra? Does every element of the group algebra have an inverse for addition?

The identity under group-algebra addition is the function z that assigns zero to every group element, i.e., $z(\sigma)=0$ for all σ. Using the definition of addition in the group algebra: $(f+z)(\sigma)=f(\sigma)+z(\sigma)=f(\sigma)+0=f(\sigma)$.

The inverse of f under addition is the function $f_{\_}$ defined by $f_{\_}(\sigma)=-f(\sigma)$. To see that it is the additive inverse: $(f+f_{\_})(\sigma)=f(\sigma)+f_{\_}(\sigma)=f(\sigma)+\left(-f(\sigma)\right)=0$.

*F. What is the identity element in the group algebra for *? Does every element of the group algebra have an inverse for *?*

The identity for * is the one-hot basis element f_e, *since* $\left(f_e * g\right)(\tau)=g(e^{-1}\tau)=g(\tau)$. However, not every element has an inverse. An example of a member of the algebra that does not have an inverse is the constant function, $c(\sigma)=1$, which can also be written as $c=\sum_{\sigma\in S} f_\sigma$, i.e., a sum over all of the one-hots. To see that $c*g$ cannot be f_e:

$c*g=\left(\sum_{\sigma\in S} f_\sigma\right)*g=\sum_{\sigma\in S}\left(f_\sigma * g\right)$, because * is linear. Then from the definition of *: $\left(c*g\right)(\tau)=\sum_{\sigma\in S}\left(f_\sigma * g\right)(\tau)=\sum_{\sigma\in S} g(\sigma^{-1}\tau)$. Now, note that as σ ranges over the group, then so does $\sigma^{-1}\tau$. So in final sum, the arguments of g sample each element of the group once, for any choice of τ. So $c*g$ is constant on the group. Since $f_e(e)=1$ but 0 elsewhere, $c*g$ cannot be f_e.

G. Let $g=\sum_{\sigma\in S} g(\sigma) f_\sigma$ and $h=\sum_{\sigma\in S} h(\sigma) f_\sigma$. Write $g*h$ as an explicit sum of one-hot basis elements.

With $g=\sum_{\sigma\in S} g(\sigma) f_\sigma$ and $h=\sum_{\sigma'\in S} h(\sigma') f_{\sigma'}$, linearity of * means that

$$g*h=\left(\sum_{\sigma\in S} g(\sigma) f_\sigma\right)*\left(\sum_{\sigma'\in S} h(\sigma') f_{\sigma'}\right)=\sum_{\sigma\in S}\sum_{\sigma'\in S} g(\sigma)h(\sigma')\left(f_\sigma * f_{\sigma'}\right).$$

Applying part B:

$$g*h=\sum_{\sigma\in S}\sum_{\sigma'\in S} g(\sigma)h(\sigma') f_{\sigma\sigma'}.$$

As σ and σ' range independently over the group, their product $\sigma\sigma'$ duplicates many values. So we collect terms for which $\sigma\sigma'=\tau$. This means that $\sigma'=\sigma^{-1}\tau$. So

$$g*h=\sum_{\tau\in S}\left(\sum_{\sigma\in S}g(\sigma)h(\sigma^{-1}\tau)\right)f_\tau.$$ That is, $\left(g*h\right)(\tau)=\sum_{\sigma\in S}g(\sigma)h(\sigma^{-1}\tau)$.

Other than a normalizing factor, this says that * is a convolution.

*Q2, Q3, and Q4 justify the specific way that * is defined. Alternative choices make either Q1A or Q1B (or both) less pretty.*

Q2. Alternative construction I.
Say $f_\sigma\circ g$ is the function on the group for which $\left(f_\sigma\circ g\right)(\tau)=g(\sigma\tau)$.

A. Show that Q1A above becomes $f_{\sigma_1}\circ\left(f_{\sigma_2}\circ g\right)=f_{\sigma_2\sigma_1}\circ g$.

Evaluate at a typical element τ:

$$\left(f_{\sigma_1}\circ\left(f_{\sigma_2}\circ g\right)\right)(\tau)=\left(f_{\sigma_2}\circ g\right)(\sigma_1\tau)=g\left(\sigma_2(\sigma_1\tau)\right)=g\left((\sigma_2\sigma_1)\tau\right)=\left(f_{\sigma_2\sigma_1}\circ g\right)(\tau)$$

B. Show that Q1B becomes $f_{\sigma_1}\circ f_{\sigma_2}=f_{\sigma_1^{-1}\sigma_2}$.

Evaluate $h=f_{\sigma_1}\circ f_{\sigma_2}$ at a typical group element τ. $h(\tau)=f_{\sigma_2}(\sigma_1\tau)$: it is zero unless $\sigma_1\tau=\sigma_2$, and 1 when equality holds. But $\sigma_1\tau=\sigma_2$ is equivalent to $\tau=\sigma_1^{-1}\sigma_2$, so $f_{\sigma_1}\circ f_{\sigma_2}=f_{\sigma_1^{-1}\sigma_2}$

Q3. Alternative construction II.
Say $f_\sigma\circ g$ is the function on the group for which $\left(f_\sigma\circ g\right)(\tau)=g(\tau\sigma^{-1})$

A. Show that Q1A above becomes $f_{\sigma_1}\circ\left(f_{\sigma_2}\circ g\right)=f_{\sigma_2\sigma_1}\circ g$.

Evaluate at a typical element τ:

$$\left(f_{\sigma_1}\circ\left(f_{\sigma_2}\circ g\right)\right)(\tau)=\left(f_{\sigma_2}\circ g\right)(\tau\sigma_1^{-1})=g\left((\tau\sigma_1^{-1})\sigma_2^{-1}\right)=g\left(\tau(\sigma_1^{-1}\sigma_2^{-1})\right)=g\left(\tau(\sigma_2\sigma_1)^{-1}\right)=\left(f_{\sigma_2\sigma_1}\circ g\right)(\tau)$$

B. Show that Q1B becomes $f_{\sigma_1}\circ f_{\sigma_2}=f_{\sigma_2\sigma_1}$.

Evaluate $h=f_{\sigma_1}\circ f_{\sigma_2}$ at a typical group element τ. $h(\tau)=f_{\sigma_2}(\tau\sigma_1^{-1})$: it is zero unless $\tau\sigma_1^{-1}=\sigma_2$, and 1 when equality holds. But $\tau\sigma_1^{-1}=\sigma_2$ is equivalent to $\tau=\sigma_2\sigma_1$, so $f_{\sigma_1}\circ f_{\sigma_2}=f_{\sigma_2\sigma_1}$.

Q4. Alternative construction III.
Say $f_\sigma\circ g$ is the function on the group for which $\left(f_\sigma\circ g\right)(\tau)=g(\tau\sigma)$

A. Show that Q1A above becomes $f_{\sigma_1}\circ\left(f_{\sigma_2}\circ g\right)=f_{\sigma_1\sigma_2}\circ g$ (i.e., is unchanged).

Evaluate at a typical element τ:

$$\left(f_{\sigma_1}\circ\left(f_{\sigma_2}\circ g\right)\right)(\tau)=\left(f_{\sigma_2}\circ g\right)(\tau\sigma_1)=g\left((\tau\sigma_1)\sigma_2\right)=g\left(\tau(\sigma_1\sigma_2)\right)=\left(f_{\sigma_1\sigma_2}\circ g\right)(\tau)$$

B. Show that Q1B becomes $f_{\sigma_1}\circ f_{\sigma_2}=f_{\sigma_2\sigma_1^{-1}}$.

Evaluate $h = f_{\sigma_1} \circ f_{\sigma_2}$ at a typical group element τ. $h(\tau) = f_{\sigma_2}(\tau\sigma_1)$: it is zero unless $\tau\sigma_1 = \sigma_2$, and 1 when equality holds. But $\tau\sigma_1 = \sigma_2$ is equivalent to $\tau = \sigma_2\sigma_1^{-1}$, so $f_{\sigma_1} \circ f_{\sigma_2} = f_{\sigma_2\sigma_1^{-1}}$.