Groups, Fields, and Vector Spaces

Homework #2 (2022-2023), Questions

Q1: The Group Algebra

Here we define a "group algebra" and see some of its basic properties.

A group algebra is the free vector space of functions on a group, along with an additional operation on the vectors that relies on the group operation. Specifically, let $S=\{e,\sigma,\tau,...\}$ be a group, and f and g are functions from S to a field k. The vector space operations are defined as before: addition of vectors, $(f+g)(s)=f(s)+g(s)$, where the addition is in k, and αf; and scalar multiplication of vectors $(\alpha f)(s)=\alpha\cdot(f(s))$ where the multiplication on the right is in the field k.

To make this into an algebra, we define the new operation for composing vector space elements, here denoted *. We define this on the "one-hot" basis for the free vector space and then extend by linearity to the whole space. Say f_σ is an element of the one-hot basis, i.e., the function on the group for which $f_\sigma(\tau)=1$ for $\tau=\sigma$ and 0 for $\tau\neq\sigma$. Then $f_\sigma * g$ is the function on the group for which $(f_\sigma * g)(\tau)=g(\sigma^{-1}\tau)$. (Q2, Q3, and Q4 show why we chose this definition, rather than multiplying by σ on the right, or not using the inverse.)

A. Show, for any element g of the group algebra, and one-hot basis elements f_{σ_1} and f_{σ_2}, that $f_{\sigma_1} *\left(f_{\sigma_2} * g\right)=f_{\sigma_1\sigma_2} * g$.

B. Show, for one-hot basis elements f_{σ_1} and f_{σ_2}, that $f_{\sigma_1} * f_{\sigma_2}=f_{\sigma_1\sigma_2}$.

C. Show that * is associative.

D. Is * commutative?

E. What is the identity element for addition in the group algebra? Does every element of the group algebra have an inverse for addition?

F. What is the identity element in the group algebra for *? Does every element of the group algebra have an inverse for *?

G. Let $g=\sum_{\sigma\in S} g(\sigma)f_\sigma$ and $h=\sum_{\sigma\in S} h(\sigma)f_\sigma$. Write $g*h$ as an explicit sum of one-hot basis elements.

Q2, Q3, and Q4 justify the specific way that * is defined. Alternative choices make either Q1A or Q1B (or both) less pretty.

Q2. Alternative construction I.

Say $f_\sigma \circ g$ is the function on the group for which $(f_\sigma \circ g)(\tau) = g(\sigma\tau)$.

A. Show that Q1A above becomes $f_{\sigma_1} \circ (f_{\sigma_2} \circ g) = f_{\sigma_2\sigma_1} \circ g$.

B. Show that Q1B becomes $f_{\sigma_1} \circ f_{\sigma_2} = f_{\sigma_1^{-1}\sigma_2}$.

Q3. Alternative construction II.

Say $f_\sigma \circ g$ is the function on the group for which $(f_\sigma \circ g)(\tau) = g(\tau\sigma^{-1})$

A. Show that Q1A above becomes $f_{\sigma_1} \circ (f_{\sigma_2} \circ g) = f_{\sigma_2\sigma_1} \circ g$.

B. Show that Q1B becomes $f_{\sigma_1} \circ f_{\sigma_2} = f_{\sigma_2\sigma_1}$.

Q4. Alternative construction III.

Say $f_\sigma \circ g$ is the function on the group for which $(f_\sigma \circ g)(\tau) = g(\tau\sigma)$

A. Show that Q1A above becomes $f_{\sigma_1} \circ (f_{\sigma_2} \circ g) = f_{\sigma_1\sigma_2} \circ g$ (i.e., is unchanged).

B. Show that Q1B becomes $f_{\sigma_1} \circ f_{\sigma_2} = f_{\sigma_2\sigma_1^{-1}}$.