Linear Systems: Black Boxes and Beyond

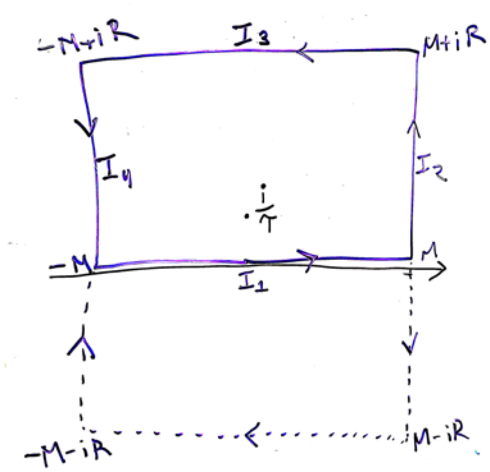
Homework #1 (2022-2023), Answers

Transfer functions and complex-analytic properties.
Q1. A simple transfer function.

For the impulse response $f(t)=\begin{cases}\frac{1}{\tau}e^{-t/\tau}, & t\ge 0\\ 0, & t<0\end{cases}$ *-- which is the impulse response of a single-stage "RC" filter with time constant --, compute the transfer function,* $\hat{f}(\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}f(t)dt$ *.*

$$\hat{f}(\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}f(t)dt=\frac{1}{\tau}\int_{0}^{\infty}e^{-i\omega t}e^{-t/\tau}dt=\frac{1}{\tau}\int_{0}^{\infty}e^{-(i\omega+1/\tau)t}dt$$
$$=\frac{1}{\tau}\frac{1}{i\omega+\frac{1}{\tau}}\left(e^{-(i\omega+1/\tau)t}\right)\Big|_{0}^{\infty}=\frac{1}{\tau}\frac{1}{i\omega+\frac{1}{\tau}}=\frac{1}{1+i\omega\tau}$$

Q2. Fourier inversion via contour integration.
For the transfer function $\hat{f}(\omega)$ *of Question 1, recover the Fourier transform* $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}\hat{f}(\omega)d\omega$ *via contour integration. Use a closed contour that runs along the real axis from, say,* $-M$ *to* M *and then returns to its start via an excursion into either the upper- or lower- half plane.*



First, take $\tau > 0$ and consider the integral $I = \int_C e^{i\omega t}\hat{f}(\omega)d\omega$ over the contour in the upper half plane, illustrated below. We first show that the integrals on the segments I_2, I_3, and I_4 approach zero as $M \to \infty$ and $R \to \infty$. This implies that, in this limit. the contour integral is equal to I_1, which is $2\pi f(t)$.

To show that $I_2 \to 0$: $I_2 = \int_0^R e^{i(M+iy)t}\hat{f}(M+iy)dy$, so

$$|I_2| \le \int_0^R \left|e^{i(M+iy)t}\right|\left|\frac{1}{1+i(M+iy)\tau}\right|dy = \left|e^{iMt}\right|\left|\int_0^R \left|e^{i(iy)t}\right|\left|\frac{1}{1+i(M+iy)\tau}\right|dy =\right.$$

$$\int_0^R e^{-yt}\left|\frac{1}{1+iM\tau-\tau^2}\right|dy = \left|\frac{1}{1+iM\tau-\tau^2}\right|\frac{1-e^{-Rt}}{t} = \frac{1-e^{-Rt}}{t}\frac{1}{\sqrt{(1-\tau^2)^2+M^2\tau^2}}$$

, which, for fixed t and τ, approaches zero as $M \to \infty$. I_4 is handled the same way, replacing M by $-M$.

To show that $I_3 \to 0$: $I_3 = \int_M^{-M} e^{i(x+iR)t}\hat{f}(x+iR)dx$, so

$$|I_3| \le \int_M^{-M} \left|e^{i(x+iR)t}\right|\left|\frac{1}{1+i(x+iR)\tau}\right|dx = e^{-Rt}\int_M^{-M}\left|e^{ixt}\right|\left|\frac{1}{1+i(x+iR)\tau}\right|dx$$

$$= e^{-Rt}\int_M^{-M}\left|\frac{1}{1+i(x+iR)\tau}\right|dx = e^{-Rt}\int_M^{-M}\left|\frac{1}{1-R\tau+ix\tau}\right|dx$$

. The final integrand is bounded away from zero, so, for fixed M, t and τ, I_3 can be made as small as desired by increasing R.

Evaluation of the contour integral: By Cauchy's Theorem, the contour integral is equal to the $2\pi i$ times sum of the residues at all the enclosed singularities. The only singularity of $\frac{1}{1+i\omega\tau}e^{i\omega t}$ is at $\omega = i/\tau$, and the residue is at that point is the limiting value of the integrand $\frac{1}{1+i\omega\tau}e^{i\omega t}$ multiplied by $\omega - i/\tau$, at $\omega = i/\tau$. But $\frac{1}{1+i\omega\tau}e^{i\omega t} = \frac{1}{i\tau}\frac{e^{i\omega t}}{\omega - i/\tau}$, so the residue is $\frac{1}{i\tau}e^{i(i/\tau)t} = \frac{1}{i\tau}e^{-t/\tau}$. So $I = 2\pi i\left(\frac{1}{i\tau}e^{-t/\tau}\right) = \frac{2\pi}{\tau}e^{-t/\tau}$. Since we showed that the other segments of the contour can be made arbitrarily small as $M \to \infty$ and $R \to \infty$, then $I_1 \to I$ in this limit. And in this limit, $I_1 = 2\pi f(t)$, so $f(t) = \frac{1}{\tau}e^{-t/\tau}$.

Note that, the critical part of the argument is that e^{-Rt} can be made small by increasing R. This argument holds for $t > 0$ since we took the contour's return path (I_3) to be in the upper half plane. For $t<0$, the magnitude along I_3 can only be controlled if it is in the lower half-plane (the dashed contour in the illustration). And in the lower half-plane, $\hat{f}(\omega)=\frac{1}{1+i\omega\tau}$ has no singularities. So Cauchy's Theorem says that the contour integral, and hence, $f(t)$, is zero when $t<0$.

Looking back at the above argument, we see that we didn't need to be able to integrate $\hat{f}(\omega)$; we just needed to know that its integral was bounded. So there's an important bottom line: when there are no singularities of $\hat{f}(\omega)$ in the lower half-plane and $\left|\hat{f}(\omega)\right|$ behaves "nicely" for large ω, then the corresponding Fourier inverse $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{f}(\omega)d\omega$ will be the impulse response of a causal system. The converse is also true: if there are singularities of $\hat{f}(\omega)$ in the lower half plane, then $\hat{f}(\omega)$ cannot be the transfer function of a causal system.

Q3. When can a linear filter be realized as a continuum concatenation of another linear filter? Consider a linear filter L of a causal system, with transfer function $\hat{L}(\omega)$.
A. If there is a linear filter B_2, for which a series combination of B_2 with itself yields L, then what is $\hat{B}_2(\omega)$? If there is a linear filter B_n, for which an series combination of n copies yields L, then what is $\hat{B}_n(\omega)$?

We need $\left(\hat{B}_2(\omega)\right)^2=\hat{L}(\omega)$, i.e., $\hat{B}_2(\omega)=\left(\hat{L}(\omega)\right)^{1/2}$. Similarly, $\left(\hat{B}_n(\omega)\right)^n=\hat{L}(\omega)$, i.e., $\hat{B}_n(\omega)=\left(\hat{L}(\omega)\right)^{1/n}$.
B. In the above scenario, as n grows, it seems reasonable to hypothesize that B_n becomes closer and closer to the identity – since the net result of n successive applications of B_n must remain fixed. What is $\hat{G}(\omega)=\lim_{n\to\infty} n(\hat{B}_n(\omega)-1)$? If this limit exists, then G can be regarded as the infinitesimal transformation that generates L, *since* $\hat{B}_n(\omega)\approx I+\frac{1}{n}\hat{G}(\omega)$.

We want $\lim_{n\to\infty} n(\left(\hat{L}(\omega)\right)^{1/n}-1)=\lim_{\varepsilon\to 0}\frac{\left(\hat{L}(\omega)\right)^{\varepsilon}-1}{\varepsilon}$. Apply L'hopital's Rule. The derivative of the denominator is 1. So we need the derivative of the numerator:

$$\hat{G}(\omega)=\lim_{n\to\infty} n(\left(\hat{L}(\omega)\right)^{1/n}-1)=\frac{d}{d\varepsilon}\left(\left(\hat{L}(\omega)\right)^{\varepsilon}-1\right)\Big|_{\varepsilon=0}=\frac{d}{d\varepsilon}\left(e^{\varepsilon\log\hat{L}(\omega)}-1\right)\Big|_{\varepsilon=0}$$

$$=\left(\log\hat{L}(\omega)e^{\varepsilon\log\hat{L}(\omega)}-1\right)\Big|_{\varepsilon=0}=\log\hat{L}(\omega)$$

. Or, $\hat{L}(\omega)=e^{\hat{G}(\omega)}$.

C. There is a converse of Q2: if there are singularities of $\hat{f}(\omega)$ in the lower half plane, then $\hat{f}(\omega)$ cannot be the transfer function of a causal system. So, given that L is a causal system (and therefore, that $\hat{L}(\omega)$ has no singularities in the lower half plane), does it follow that every causal system has a causal infinitesimal? If not, what is an example?

No. Since $\hat{G}(\omega) = \log \hat{L}(\omega)$, we need to check if the absence of singularities for $\hat{L}(\omega)$ in the lower half plane guarantees that there are no singularities for $\log \hat{L}(\omega)$. The logarithm has a singularity when its argument is either infinity or zero. So if $\hat{L}(\omega)$ has a zero in the lower half plane – which is not a singularity -- then $\hat{G}(\omega) = \log \hat{L}(\omega)$ has a singularity.

A simple example is $\hat{h}(\omega) = \dfrac{1 - i\omega\tau}{1 + i\omega\tau}$, which is related to the transfer function in Q1 by $\hat{h}(\omega) = 2\hat{f}(\omega) - 1$. $\hat{h}(\frac{-i}{\tau}) = 0$, so, although $\hat{h}(\omega)$ does not have a singularity in the lower half plane, $\log \hat{h}(\omega)$ does.

Comment: Transfer functions that have causal infinitesimals – equivalently, that do not have zeros in the lower half plane – are called "minimum phase" transfer functions.

$\hat{h}(\omega) = \dfrac{1 - i\omega\tau}{1 + i\omega\tau}$ is the archetype of a transfer function that is not minimum-phase. Note that $\left|\hat{h}(\omega)\right| = 1$ -- so that concatenation with this filter adds a phase shift, but without changing amplitudes.