Linear Systems: Black Boxes and Beyond

Homework #2 (2022-2023), Answers

Spectral Leakage

Q1. As mentioned, the amount of spectral leakage associated with a given window function $W(t)$ *can be characterized by* $\left|\tilde{W}(\Delta\omega)\right|^2$*, where* $\Delta\omega = \omega - \omega_0$*,* ω_0 *is the frequency of a infinitesimally narrow spectral peak, and* ω *is the center of a bin of the estimated power spectrum. Here we determine the behavior of* $\left|\tilde{W}(\Delta\omega)\right|^2$ *for some simple and popular window functions.*

A. For the "square" window $W_{square}(t) = \begin{cases} 1, & |t| \le \dfrac{L}{2} \\ 0, & |t| > \dfrac{L}{2} \end{cases}$*, determine* $\left|\tilde{W}_{square}(\Delta\omega)\right|^2$*, its behavior for large* $|\Delta\omega|$*, and its zeroes.*

$$\begin{aligned}\tilde{W}_{square}(\Delta\omega) &= \int_{-\infty}^{\infty} W_{square}(t)e^{-i(\Delta\omega)t}dt = \int_{-L/2}^{L/2} e^{-i(\Delta\omega)t}dt = \frac{1}{-i(\Delta\omega)}e^{-i(\Delta\omega)t}\Big|_{-L/2}^{L/2} \\ &= \frac{e^{-i(\Delta\omega)L/2} - e^{i(\Delta\omega)L/2}}{-i(\Delta\omega)} = \frac{2}{\Delta\omega}\frac{e^{i(\Delta\omega)L/2} - e^{-i(\Delta\omega)L/2}}{2i} = \frac{2}{\Delta\omega}\sin\left(\frac{(\Delta\omega)L}{2}\right)\end{aligned}.$$

It's convenient (and traditional) to write this in the form $\tilde{W}_{square}(\Delta\omega) = \dfrac{2}{\Delta\omega}\sin\left(\dfrac{(\Delta\omega)L}{2}\right) = L\,\mathrm{sinc}\left(\dfrac{(\Delta\omega)L}{2}\right)$,where $\mathrm{sinc}\,u = \dfrac{\sin u}{u}$. Note that $\mathrm{sinc}(0) = 1$ and, for large $|u|$, $\mathrm{sinc}\,u$ oscillates between $\pm\dfrac{1}{u}$. So then $\left|\tilde{W}_{square}(\Delta\omega)\right|^2 = L^2\,\mathrm{sinc}^2\left(\dfrac{(\Delta\omega)L}{2}\right)$. So for large $|\Delta\omega|$, the spectral leakage is proportional to $|\Delta\omega|^{-2}$.

Regarding the zeros: $\mathrm{sinc}(n\pi) = 0$ for any integer $n \ne 0$,and in particular, for $n = 1$. So we see that $\left|\tilde{W}_{square}(\Delta\omega)\right|^2$ has its first zero at $\dfrac{(\Delta\omega)L}{2} = \pm\pi$, i.e., at $\Delta\omega = \pm\dfrac{2\pi}{L}$, and has zeroes at every nonzero integer multiple of $\dfrac{2\pi}{L}$.

B. As in A, but for the "triangle" window $W_{triangle}(t) = \begin{cases} 1 - \dfrac{2}{L}|t|, & |t| \le \dfrac{L}{2} \\ 0, & |t| > \dfrac{L}{2} \end{cases}$.

Note that the triangular window is proportional to a convolution of the square window with itself, provided that the square window is of half the length in A (which we denote $W_{square}(t;L/2)$).

This is easier to see geometrically than analytically, but analytically:

$$\left(W_{square} * W_{square}\right)(t;L/2) = \int_{-\infty}^{\infty} W_{square}(t-\tau;L/2) W_{square}(\tau;L/2)d\tau$$
$$= \int_{\max(-L/4,t-L/4)}^{\min(L/4,L/4+t)} d\tau$$
, where the limits of integration are determined from the intersection of $-L/4 \le \tau \le L/4$ and $-L/4 \le t-\tau \le L/4$. This interval has length $\max(\frac{L}{2}-|t|,0)$, so

$$\left(W_{square} * W_{square}\right)(t;L/2) = \max(\frac{L}{2}-|t|,0) = \frac{L}{2}\max(1-\frac{2}{L}|t|,0)$$, and

$$W_{triangle}(t;L) = \frac{2}{L}\left(W_{square} * W_{square}\right)(t;L/2).$$

Now, using the fact that a convolution in the time domain corresponds to multiplication in the frequency domain, it follows that

$$\tilde{W}_{triangle}(\Delta\omega) = \frac{2}{L}\left(\tilde{W}_{square}(\Delta\omega;L/2)\right)^2 = \frac{2}{L}\left(\frac{L}{2}\right)^2 \operatorname{sinc}^2\left(\frac{(\Delta\omega)L}{4}\right) = \frac{L}{2}\operatorname{sinc}^2\left(\frac{(\Delta\omega)L}{4}\right).$$

So, for large $|\Delta\omega|$, the spectral leakage $\left|\tilde{W}_{triangle}(\Delta\omega)\right|^2$ is proportional to $|\Delta\omega|^{-4}$. However, note also that this more rapid decline in spectral leakage (compared to $\left|\tilde{W}_{square}(\Delta\omega)\right|^2$) is at a cost: the zeroes of $\left|\tilde{W}_{triangle}(\Delta\omega)\right|^2$ are at the zeros of $\frac{(\Delta\omega)(L/2)}{2} = \pm n\pi$ (for $n \ne 0$), i.e., at $\Delta\omega = \pm\frac{4n\pi}{L}$.

C. As in A, but for the "cosine bell" window $W_{cosbell}(t) = \begin{cases} \frac{1}{2}\left(1+\cos\frac{2\pi t}{L}\right), & |t| \le \frac{L}{2} \\ 0, & |t| > \frac{L}{2} \end{cases}$.

$$\tilde{W}_{cosbell}(\Delta\omega) = \int_{-\infty}^{\infty} W_{cosbell}(t)e^{-i(\Delta\omega)t}dt = \frac{1}{2}\int_{-L/2}^{L/2}\left(1+\cos\frac{2\pi t}{L}\right)e^{-i(\Delta\omega)t}dt$$
$$= \frac{1}{2}\int_{-L/2}^{L/2}\left(1+\frac{1}{2}e\left(\frac{2\pi i}{L}t + e^{\frac{-2\pi i}{L}t}\right)\right)e^{-i(\Delta\omega)t}dt$$
.
$$= \frac{1}{2}\int_{-L/2}^{L/2} e^{-i(\Delta\omega)t}dt + \frac{1}{4}\int_{-L/2}^{L/2} e^{-\left(\Delta\omega-\frac{2\pi}{L}\right)it}dt + \frac{1}{4}\int_{-L/2}^{L/2} e^{-\left(\Delta\omega+\frac{2\pi}{L}\right)it}dt$$

Each of these integrals was evaluated in part A, so:

$$\tilde{W}_{cosbell}(\Delta\omega) = \frac{1}{2}\tilde{W}_{square}(\Delta\omega) + \frac{1}{4}\tilde{W}_{square}(\Delta\omega - \frac{2\pi}{L}) + \frac{1}{4}\tilde{W}_{square}(\Delta\omega + \frac{2\pi}{L})$$

$$= \frac{1}{2}L\,\mathrm{sinc}\left(\frac{(\Delta\omega)L}{2}\right) + \frac{1}{4}L\,\mathrm{sinc}\left(\frac{\left(\Delta\omega - \frac{2\pi}{L}\right)L}{2}\right) + \frac{1}{4}L\,\mathrm{sinc}\left(\frac{\left(\Delta\omega + \frac{2\pi}{L}\right)L}{2}\right)$$

$$= \frac{1}{2}L\,\mathrm{sinc}\left(\frac{(\Delta\omega)L}{2}\right) + \frac{1}{4}L\,\mathrm{sinc}\left(\frac{(\Delta\omega)L}{2} - \pi\right) + \frac{1}{4}L\,\mathrm{sinc}\left(\frac{(\Delta\omega)L}{2} + \pi\right)$$

Regarding asymptotic behavior: Put $u = \frac{(\Delta\omega)L}{2}$. Then

$$\tilde{W}_{cosbell}(\Delta\omega) = \frac{L}{2}\frac{\sin(u)}{u} + \frac{L}{4}\frac{\sin(u-\pi)}{u-\pi} + \frac{L}{4}\frac{\sin(u+\pi)}{u+\pi}$$

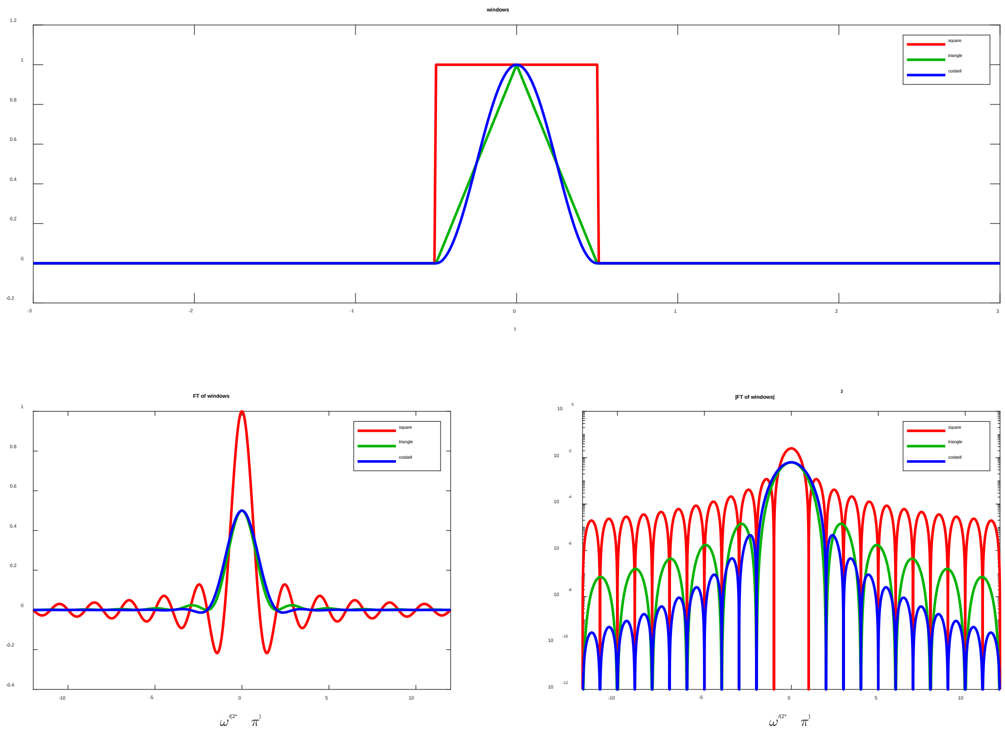
$$= \frac{L}{4}\sin(u)\left(\frac{2}{u} - \frac{1}{u-\pi} - \frac{1}{u+\pi}\right) = \frac{L}{4u(u-\pi)(u+\pi)}\sin(u)\left(2(u-\pi)(u+\pi) - u(u+\pi) - u(u-\pi)\right)$$

$$= \frac{L}{4u(u-\pi)(u+\pi)}\sin(u)\left(-2\pi^2\right)$$

So for large $|\Delta\omega|$, $\tilde{W}_{cosbell}(\Delta\omega)$ has maxima and minima proportional to $|\Delta\omega|^{-3}$, and the spectral leakage $\left|\tilde{W}_{cosbell}(\Delta\omega)\right|^2$ is proportional to $|\Delta\omega|^{-6}$.

Regarding zeros (see part A): the first term has zeros when $\Delta\omega$ is equal to all nonzero multiplies of $\frac{2\pi}{L}$; the second term has zeros when $\Delta\omega$ is equal to all integer multiplies of $\frac{2\pi}{L}$ except $\frac{2\pi}{L}$; the third term has zeros when $\Delta\omega$ is equal to all integer multiplies of $\frac{2\pi}{L}$ except $-\frac{2\pi}{L}$. So there are zeros at $\frac{2\pi}{L}n$ for integers n except $n \in \{-1, 0, 1\}$.

D. Plot the windows and their corresponding spectral leakage.



Q2. Algebraic properties of time- and frequency-domain restriction
Consider the vector space of square-integrable functions of time (our standard Hilbert space), and the standard inner product, $\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\overline{\tilde{g}(\omega)}d\omega$ *(the last equality from Parseval's Theorem). Now consider a set of times* S_{time} *and an arbitrary domain of (real-valued) frequencies* S_{freq}.

Define two linear operators: D, *defined by* $Df(x) = \begin{cases} f(x), & x \in S_{time} \\ 0, & x \notin S_{time} \end{cases}$, *and* B, *defined by its action on the Fourier transform of* f: $B\tilde{f}(\omega) = \begin{cases} \tilde{f}(\omega), & \omega \in S_{freq} \\ 0, & \omega \notin S_{freq} \end{cases}$. *In the standard development of multitaper analysis,* S_{time} *is an interval, and* S_{freq} *is a range such as* $|\omega| \le \omega_{max}$; *here we are dispensing with this requirement and just focusing on the algebraic properties.*

A. Show that D *and* B *are self-adjoint.*
For D: $\langle Df,g\rangle = \int_{-\infty}^{\infty} Df(x)\overline{g(x)}dx = \int_{S_{time}} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} f(x)\overline{Dg(x)}dx = \langle Df,g\rangle$,
and similarly for B:

$$\left\langle Bf, g \right\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} B\tilde{f}(\omega)\overline{\tilde{g}(\omega)}d\omega = \frac{1}{2\pi}\int_{S_{freq}} \tilde{f}(\omega)\overline{\tilde{g}(\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\overline{B\tilde{g}(\omega)}d\omega = \left\langle f, Bg \right\rangle.$$

B. Show that that D and B are projections.

Directly from the definitions, $D^2 = D$ and $B^2 = B$. So D and B are self-adjoint and idempotent, so they are projections.

C. Do D and B commute?

No. Consider, for example, S_{time} and S_{freq} to be intervals. BDf has finite bandwidth, and therefore, must have an infinite duration over which it is nonzero (there are no high frequencies that would enable it to cut to exactly zero). But DBf is restricted to an an interval. So, in particular, DB and BD are not projections.

D. Show that DBD and BDB are self-adjoint.

Using A:

$\left\langle DBDf, g \right\rangle = \left\langle BDf, Dg \right\rangle = \left\langle Df, BDg \right\rangle = \left\langle f, DBDg \right\rangle$, and similarly for BDB.

E. From D, we see that DBD and BDB are "normal" operators (they commute with their adjoints), and therefore, via the spectral theorem, their eigenvectors span the space. Show that eigenvalues of DBD are also eigenvalues of DB, and that if f is an eigenvector of DBD, then Df is an eigenvector of DB, with the same eigenvalue. Similarly, if *f is an eigenvector of BDB, then Bf is an eigenvector of BD, with the same eigenvalue.*

Say $DBDf = \lambda f$. Then, $D^2BDf = D(\lambda f) = \lambda(Df)$. On the other hand, since D is idempotent, $D^2BD = DBD$. So $DBDf = \lambda(Df)$.

Note that, since the eigenvectors $\left\{ f \right\}$ of DBD (a normal operator) form an orthonormal basis for the whole space, the projections $\{Df\}$ form a basis for functions that are only nonzero on S_{time}. Basis functions corresponding to different eigenvalues are orthonormal – since D is self-adjoint. Put another way, the functions $\left\{ f \right\}$ are orthonormal on the entire domain, and also on the domain S_{time}. This is sometimes called "double orthogonality."

F. Show that, for any f in the vector space, that $\left\langle Df, Df \right\rangle \le \left\langle f, f \right\rangle$ and similarly $\left\langle Bf, Bf \right\rangle \le \left\langle f, f \right\rangle$ all eigenvalues of D are ≤ 1. Similarly for B.

$\left\langle Df, Df \right\rangle = \int_{-\infty}^{\infty} Df(x)\overline{Df(x)}dx = \int_{S_{time}} f(x)\overline{f(x)}dx \le \int_{-\infty}^{\infty} f(x)\overline{f(x)}dx = \left\langle f, f \right\rangle$. Similarly for B.

(Typiccally this inequality is strict, i.e., if S_{time} is a proper subset.)

G. Using F, show that all eigenvalues of DB *are* ≤ 1. Say $DBf = \lambda f$. Then $\lambda^2 \langle f, f \rangle = \langle DBf, DBf \rangle \leq \langle Bf, Bf \rangle \leq \langle f, f \rangle$, so $\lambda^2 \leq 1$.