Linear Systems: Black Boxes and Beyond

Homework #2 (2022-2023), Questions

Spectral Leakage

Q1. As mentioned, the amount of spectral leakage associated with a given window function $W(t)$ can be characterized by $\left|\tilde{W}(\Delta\omega)\right|^2$, where $\Delta\omega = \omega - \omega_0$, ω_0 is the frequency of a infinitesimally narrow spectral peak, and ω is the center of a bin of the estimated power spectrum. Here we determine the behavior of $\left|\tilde{W}(\Delta\omega)\right|^2$ for some simple and popular window functions.

A. For the "square" window $W_{square}(t) = \begin{cases} 1, & |t| \le \frac{L}{2} \\ 0, & |t| > \frac{L}{2} \end{cases}$, determine $\left|\tilde{W}_{square}(\Delta\omega)\right|^2$, its behavior for large $|\Delta\omega|$, and its zeroes.

B. As in A, but for the "triangle" window $W_{triangle}(t) = \begin{cases} 1 - \frac{2}{L}|t|, & |t| \le \frac{L}{2} \\ 0, & |t| > \frac{L}{2} \end{cases}$.

C. As in A, but for the "cosine bell" window $W_{cosbell}(t) = \begin{cases} \frac{1}{2}\left(1 + \cos\frac{2\pi t}{L}\right), & |t| \le \frac{L}{2} \\ 0, & |t| > \frac{L}{2} \end{cases}$.

D. Plot the windows and their corresponding spectral leakage.

Q2. Algebraic properties of time- and frequency-domain restriction
Consider the vector space of square-integrable functions of time (our standard Hilbert space), and the standard inner product, $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\overline{\tilde{g}(\omega)}d\omega$ (the last equality from Parseval's Theorem). Now consider a set of times S_{time} and an arbitrary domain of (real-valued) frequencies S_{freq}.

Define two linear operators: D, defined by $Df(x) = \begin{cases} f(x), & x \in S_{time} \\ 0, & x \notin S_{time} \end{cases}$, and B, defined by its action on the Fourier transform of f: $B\tilde{f}(\omega) = \begin{cases} \tilde{f}(\omega), & \omega \in S_{freq} \\ 0, & \omega \notin S_{freq} \end{cases}$. In the standard development of multitaper analysis, S_{time} is an interval, and S_{freq} is a range such as $|\omega| \le \omega_{max}$; here we are dispensing with this requirement and just focusing on the algebraic properties.

A. Show that D and B are self-adjoint.

B. Show that that D and B are projections.

C. Do D and B commute?

D. Show that DBD and BDB are self-adjoint.

E. From D, we see that DBD and BDB are "normal" operators (they commute with their adjoints), and therefore, via the spectral theorem, their eigenvectors span the entire vector space. Show that eigenvalues of DBD are also eigenvalues of DB, and that if f is an eigenvector of DBD, then Df is an eigenvector of DB, with the same eigenvalue. Similarly, if f is an eigenvector of BDB, then Bf is an eigenvector of BD, with the same eigenvalue.

F. Show that, for any f in the vector space, that $\langle Df, Df \rangle \le \langle f, f \rangle$ and similarly $\langle Bf, Bf \rangle \le \langle f, f \rangle$.

G. Using F, show that all eigenvalues of DB (and BD) are ≤ 1.