Exam, 2022-2023 Questions and Solutions

Note that many of the answers are far more detailed than required for full credit.

1. Group theory: intrinsically-defined subgroups

Here we construct two important intrinsically-defined subgroups in any group.

A. *For a group G, its commutator $D(G)$ is defined by the set of all elements $[x,y]=x^{-1}y^{-1}xy$, along with all elements generated by products of such elements. Show that the commutator is a subgroup.*
Identity: $D(G)$ contains the identity, since $[x,x]=x^{-1}x^{-1}xx=e$ for any $x\in G$.
Inverse: $[x,y]^{-1}=\left(x^{-1}y^{-1}xy\right)^{-1}=y^{-1}x^{-1}yx=[y,x]^{-1}$, so any of the generators $[x,y]$ of $D(G)$ also have their inverses in $D(G)$.
Associativity: Since $D(G)$ is a (perhaps proper) subset of G, this is inherited from associativity of G.

B. *Show that the commutator is a normal subgroup.*
We need to show that if $a\in D(G)$ and $g\in G$, then $g^{-1}ag\in D(G)$. It suffices to show that this holds for a typical generator $a=[x,y]$, since if $b=a_1...a_k$, a product of such generators, then $g^{-1}bg=g^{-1}a_1a_2...a_kg=\left(g^{-1}a_1g\right)\left(g^{-1}a_2g\right)...\left(g^{-1}a_kg\right)$.

For $a=[x,y]$,
$$\begin{aligned} g^{-1}ag &= g^{-1}[x,y]g=g^{-1}\left(x^{-1}y^{-1}xy\right)g=g^{-1}x^{-1}gg^{-1}y^{-1}gg^{-1}xgg^{-1}yg \\ &=\left(g^{-1}x^{-1}g\right)\left(g^{-1}y^{-1}g\right)\left(g^{-1}xg\right)\left(g^{-1}yg\right) \\ &=\left(g^{-1}xg\right)^{-1}\left(g^{-1}yg\right)^{-1}\left(g^{-1}xg\right)\left(g^{-1}yg\right) \\ &=[g^{-1}xg,g^{-1}yg]\end{aligned}$$
, showing that $g^{-1}ag\in D(G)$.

C. *The center of a group $Z(G)$ is defined as the subset of all elements that commute with all elements of G. Show that the center is a subgroup.*
Identity: $Z(G)$ contains the identity, since the identity commutes with all $g\in G$.
Inverse: If x commutes with all $g\in G$, then so does x^{-1}, since $gx=xg\Leftrightarrow x=g^{-1}xg\Leftrightarrow xg^{-1}=g^{-1}x$.
(First step is left multiplication by g^{-1}, second step is right multiplication by g^{-1})

D. *Show that the center is a normal subgroup.*
We need to show that if $a\in Z(G)$ and $g\in G$, then $g^{-1}ag\in Z(G)$. So say that a commutes with all $x\in G$. Then it also commutes with any $g^{-1}xg$. That is, for any $x\in G$, $a\left(g^{-1}xg\right)=\left(g^{-1}xg\right)a$. But
$$\begin{aligned} &ag^{-1}xg=g^{-1}xga\Leftrightarrow \\ &ag^{-1}x=g^{-1}xgag^{-1}\Leftrightarrow \\ &gag^{-1}x=xgag^{-1}\Leftrightarrow \\ &\left(gag^{-1}\right)x=x\left(gag^{-1}\right)\end{aligned}$$
. (First step: right multiplication by g^{-1}, second step: left multiplication by g)
The final step shows that $g^{-1}ag$ commutes with any $x\in G$, and hence, is in the center.

E. *For $SO(n)$, the group of rotations in an n-dimensional Euclidean space, what is the commutator subgroup ? Demonstrate by displaying a generator of the commutator group by computing $[x,y]$ for* *group elements x and y that are close to the identity, but do not commute. An approximate argument suffices.*

$n=2$: Since $SO(2)$ is commutative, the commutator subgroup contains only the identity.

$n\geq 3$: Choose three orthogonal axes. Let $x(\delta)$ and $y(\delta)$ correspond to small rotations in planes that share one axis, and have the other axis orthogonal. For example,

$x(\delta)=\begin{pmatrix}\cos\delta & \sin\delta & 0\\ -\sin\delta & \cos\delta & 0\\ 0 & 0 & 1\end{pmatrix}=I+M_{12}\delta+\frac{1}{2}{M_{12}}^2\delta^2+O(\delta^3)$, where $M_{12}=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, and

$y(\delta)=\begin{pmatrix}\cos\delta & 0 & \sin\delta\\ 0 & 1 & 0\\ -\sin\delta & 0 & \cos\delta\end{pmatrix}=I+M_{13}\delta+\frac{1}{2}{M_{13}}^2\delta^2+O(\delta^2)$, where $M_{13}=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}$. (This is justified, for example, by Taylor expansion). Below we also use M_{jk} in to denote a matrix with $+1$ in row j, column k and -1 in row k, column j.

Now calculate $[x(\delta),y(\delta)]$, noting that $\left(x(\delta)\right)^{-1}=x(-\delta)$ and $\left(y(\delta)\right)^{-1}=y(-\delta)$.

$$\begin{aligned}
&[x(\delta),y(\delta)]=\left(x(\delta)\right)^{-1}\left(y(\delta)\right)^{-1}x(\delta)y(\delta)\\
&=\left(I-M_{12}\delta+\frac{1}{2}{M_{12}}^2\delta^2\right)\left(I-M_{13}\delta+\frac{1}{2}{M_{13}}^2\delta^2\right)\left(I+M_{12}\delta+\frac{1}{2}{M_{12}}^2\delta^2\right)\left(I+M_{13}\delta+\frac{1}{2}{M_{13}}^2\delta^2\right)+O(\delta^3)\\
&=I+\left(-M_{12}-M_{13}+M_{12}+M_{13}\right)\delta+\\
&\left(\left(\frac{1}{2}+\frac{1}{2}-1\right){M_{12}}^2+\left(\frac{1}{2}+\frac{1}{2}-1\right){M_{13}}^2+\left(1-1+1\right)M_{12}M_{13}+\left(-1\right)M_{13}M_{12}\right)\delta^2+O(\delta^3)\\
&=I+\left(M_{12}M_{13}-M_{13}M_{12}\right)\delta^2+O(\delta^3)
\end{aligned}$$

$$M_{12}M_{13}-M_{13}M_{12}=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}-\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}-\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & -1 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}=M_{32}$$

. So

$[x(\delta),y(\delta)]=\delta z(\delta)+O(\delta^3)$, where $z(\delta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\delta & -\sin\delta\\ 0 & \sin\delta & \cos\delta\end{pmatrix}=I+M_{32}\delta+\frac{1}{2}{M_{32}}^2\delta^2+O(\delta^2)$. That is, $[x(\delta),y(\delta)]$ is a rotation of size δ^2 around the axis shared by $x(\delta)$ and $y(\delta)$.

Since this shared axis can be any axis, elements of the form $[x(\delta),y(\delta)]$ can be small rotations about any axis, and therefore, generate all of $SO(n)$.

F. *For $SO(n)$, use the analysis in E to determine the center.*

$n=2$: $SO(2)$ is the group of rotations about an axis. This is commutative, so the center is the entire group. For $n \ge 3$, any element can be considered a product of rotations in pairs of orthogonal planes. So a small group element $g(\delta)$ can be considered to be a product of small rotations in such pairs of planes,

$$g(\delta)=\left(I+a_{12}M_{12}\delta+\frac{1}{2}a_{12}^2M_{12}^2\delta^2\right)\left(I+a_{34}M_{34}\delta+\frac{1}{2}a_{34}^2M_{34}^2\delta^2\right)\left(I+a_{56}M_{56}\delta+\frac{1}{2}a_{56}^2M_{56}^2\delta^2\right)...+O(\delta^3)$$

e.g., $=I+\left(a_{12}M_{12}+a_{34}M_{34}+a_{56}M_{56}..\right)\delta+$

$$\left(\frac{1}{2}a_{12}^2M_{12}^2+\frac{1}{2}a_{34}^2M_{34}^2+\frac{1}{2}a_{56}^2M_{56}^2+...+a_{12}a_{34}M_{12}M_{34}+a_{12}a_{56}M_{12}M_{56}+...\right)\delta^2+O(\delta^3).$$

Compute $[g(\delta),y(\delta)]$ up to $O(\delta^2)$, At $O(\delta)$, all terms cancel. At $O(\delta^2)$, the terms like $M_{st}M_{uv}$ (all subscripts distinct) cancel. The only terms that do not drop out are those that refer to planes in the expansion of $g(\delta)$ that intersect the plane in which $y(\delta)$ rotates.

So

$$[g(\delta),y(\delta)]=I+a_{12}\left(M_{12}M_{13}-M_{13}M_{12}\right)\delta^2+a_{34}\left(M_{34}M_{13}-M_{13}M_{34}\right)\delta^2+O(\delta^3)$$
$$=I+a_{12}M_{32}\delta^2+a_{34}M_{41}\delta^2+O(\delta^3).$$

For g to be in the center, it has to commute with any y, so this would have to be the identity. For y in the (1,3) plane, this can only happen a_{12} and a_{34} are zero. But y could be chosen to interact with any of the planes in which g acts as a pure rotation – so all of the coefficients a_{uv} must be zero, and g must be the identity.

2. Fourier analysis as a unitary transformation

In its standard form, Fourier transformation is almost a unitary transformation – dot-product of two functions differs from that of their Fourier transforms by a factor of 2π (Parseval's Theorem). We can make it unitary by a slightly nonstandard formulation, which presents a nearly symmetric relationship between complex-valued functions on the line and their Fourier transforms:

$$\hat{f}(x)=\left(Sf\right)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixu}f(u)du \qquad (1)$$

and

$$f(x)=\left(S^{-1}\hat{f}\right)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixu}\hat{f}(u)du. \qquad (2)$$

In this formulation, Fourier transformation is truly unitary: $\int_{-\infty}^{\infty}\hat{f}(x)\overline{\hat{g}(x)}dx=\int_{-\infty}^{\infty}f(x)\overline{g(x)}dx$. *Here we write Fourier transformation as an operator (i.e.,* $\hat{f}(x)=\left(Sf\right)(x)$*), to emphasize this viewpoint.*

A. *What is* $\left(S^2f\right)(x)$*? What is* $\left(S^4f\right)(x)$*? (Hint: Consider* Sg *for* $g(x)=f(-x)$*.)*

$$\left(Sg\right)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixu}f(-u)du=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixu}f(u)du=\left(S^{-1}f\right)(x),$$

(first equality is eq 1 above, then change variables, then equation 2 above)

So, if $Sg = S^{-1}f$, then $S^2 g = f$, i.e., $\left(S^2 g\right)(x) = f(x) = g(-x)$.
And $\left(S^4 g\right)(x) = \left(S^2(S^2 g)\right)(x) = \left(S^2 g\right)(-x) = g(x)$

B. *What are the possible eigenvalues of S ?*
If h is an eigenfunction of S with eigenvalue λ, then in general $S^n h = \lambda^n h$. Since, for any h, $S^4 h = h$ (part A), then $\lambda^4 = 1$. So, the possible values of λ are $\{1, -1, i, -i\}$.

C. *Find an eigenvector for the eigenvalue of largest real part. (Hint: consider Gaussians.)*
For eigenvalue $\lambda = 1$, we need a function that is preserved under Fourier transformation. Consider a Gaussian of variance V: $g(x;V) = \dfrac{1}{\sqrt{2\pi V}}\exp(-x^2/2V)$ (which is properly normalized). Then (completing the square)

$$\left(Sg\right)(x;V) = \frac{1}{\sqrt{2\pi V}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{2V}\right)\exp(-ixu)\,du$$

$$= \frac{1}{\sqrt{2\pi V}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2 V}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{2V}\right)\exp(-ixu)\exp\left(\frac{x^2 V}{2}\right)du\,.$$

$$= \frac{1}{\sqrt{2\pi V}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2 V}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\left(\frac{u}{\sqrt{V}} + ix\sqrt{V}\right)^2\right)du$$

Change of variables, $t = u + ixV$:

$$\left(Sg\right)(x;V) =$$

$$= \frac{1}{\sqrt{2\pi V}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2 V}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{t^2}{2V}\right)dt$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2 V}{2}\right)\frac{1}{\sqrt{2\pi V}}\int_{-\infty}^{\infty}\exp\left(-\frac{t^2}{2V}\right)dt$$

Recognizing that the last two terms are a properly-normalized Gaussian of variance V:

$$\left(Sg\right)(x;V) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2 V}{2}\right).$$

So, for $h_1(x) = g(x;V)$ for $V = 1$ (i.e., a unit-variance Gaussian), $Sh_1 = h_1$.

D. *Find an eigenvector for the eigenvalue whose real part is zero. (Hint: Consider $S(f')$.)*

We need an eigenfunction for $\lambda = i$ or $\lambda = -i$. Generically, from eq. (1), integrating by parts:

$$\left(Sf'\right)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixu}\frac{d}{du}f(u)\,du = \frac{1}{\sqrt{2\pi}}f(x)e^{-ixu}\Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-ix)e^{-ixu}f(u)\,du\,.$$

Considering functions f that approach zero in an integrable fashion for arguments $\to \pm\infty$, this is

$$\left(Sf'\right)(x) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-ix)e^{-ixu}f(u)\,du = -ix\left(Sf\right)(x)\,.$$

Now take $f = h_1(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2V}$ from part C. So $Sf = f$. And,

$(Sf')(x) = -ix(Sf)(x) = -ixf(x)$, but also $f'(x) = -\frac{x}{\sqrt{2\pi}}e^{-x^2/2} = -xf(x)$. So $Sf' = if'$, and $f'(x) = -\frac{x}{\sqrt{2\pi}}e^{-x^2/2}$ has eignenvalue i.

E. Find an eigenvector for the eigenvalue of smallest real part. (Hint: Consider $S(f'')$.)

We need an eigenfunction for $\lambda = -1$. Applying the first result of D twice:

$$(Sf'')(x) = -ix(Sf')(x) = -x^2(Sf)(x).$$

Again take $f = h_1(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2V}$ from part C, so $Sf = f$. $f''(x) = \frac{d}{dx}\left(-\frac{x}{\sqrt{2\pi}}e^{-x^2/2}\right) = (x^2-1)f(x)$.

So $-f(x) - f''(x) = -x^2 f(x) = -x^2(Sf)(x) = (Sf'')(x)$. So S preserves the two-dimensional subspace spanned by $f(x)$ and $f''(x)$:

$$\begin{cases} Sf = f \\ Sf'' = -f - f'' \end{cases}, \text{ and acts like the transformation } \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \text{ which has an eigenvalue of} -1.$$

To find it: we seek some $k = f'' + \alpha f$ for which $Sk = -k$. $Sk = Sf'' + \alpha Sf = -f - f'' + \alpha f$, so we need $-f - f'' + \alpha f = -(\alpha f + f'')$, i.e., $\alpha = \frac{1}{2}$.

Finally, $k = f'' + \alpha f = f'' + \frac{1}{2}f = \frac{1}{\sqrt{2\pi}}\left(x^2 - \frac{1}{2}\right)e^{-x^2/2}$ has eigenvalue -1