Exam, 2022-2023 Questions

Do any 8 subparts (out of the 6 in Q1 and 5 in Q2).

1. Group theory: intrinsically-defined subgroups

Here we construct two important intrinsically-defined subgroups in any group.

A. For a group G, its commutator $D(G)$ is defined by the set of all elements $[x,y]=x^{-1}y^{-1}xy$, along with all elements generated by products of such elements. Show that the commutator is a subgroup.
B. Show that the commutator is a normal subgroup.
C. The center of a group $Z(G)$ is defined as the subset of all elements that commute with all elements of G. Show that the center is a subgroup.
D. Show that the center is a normal subgroup.
E. For $SO(n)$, the group of rotations in an n-dimensional Euclidean space, what is the commutator subgroup ? Demonstrate by displaying a generator of the commutator group by computing $[x,y]$ for group elements x and y that are close to the identity, but do not commute. An approximate argument suffices.
F. For $SO(n)$, use the analysis in E to determine the center.

2. Fourier analysis as a unitary transformation

In its standard form, Fourier transformation is almost a unitary transformation – dot-product of two functions differs from that of their Fourier transforms by a factor of 2π (Parseval's Theorem). We can make it unitary by a slightly nonstandard formulation, which presents a nearly symmetric relationship between complex-valued functions on the line and their Fourier transforms:

$$\hat{f}(x)=\left(Sf\right)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixu}f(u)du \qquad (1)$$

and

$$f(x)=\left(S^{-1}\hat{f}\right)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixu}\hat{f}(u)du\,. \qquad (2)$$

In this formulation, Fourier transformation is truly unitary: $\int_{-\infty}^{\infty}\hat{f}(x)\overline{\hat{g}(x)}dx=\int_{-\infty}^{\infty}f(x)\overline{g(x)}dx$. Here we write Fourier transformation as an operator (i.e., $\hat{f}(x)=\left(Sf\right)(x)$), to emphasize this viewpoint.

A. What is $\left(S^2f\right)(x)$? What is $\left(S^4f\right)(x)$? (Hint: Consider Sg for $g(x)=f(-x)$.)
B. What are the possible eigenvalues of S ?
C. Find an eigenvector for the eigenvalue of largest real part. (Hint: consider Gaussians.)
D. Find an eigenvector for the eigenvalue whose real part is zero. (Hint: consider $S(f')$.)
E. Find an eigenvector for the eigenvalue of smallest real part. (Hint: consider $S(f'')$.)