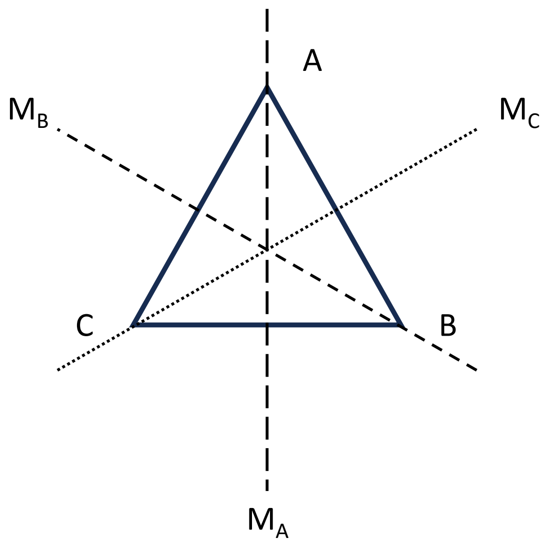
Groups, Fields, and Vector Spaces

Homework #1 (2024-2025), Answers

Q1: Multiple views of the same group: rotations and reflections of the triangle.



Consider the rotations and reflections of an equilateral triangle. Designate the identity transformation by I, a clockwise rotation of $\frac{2\pi}{3}$ *by R, a counter-clockwise rotation of* $\frac{2\pi}{3}$ *by S, and the three mirror reflections (as diagrammed here) by* M_A, M_B, *and* M_C.

i. *Write out how these group elements act, viewing each group element x as the permutation* $\pi(x)$ *that maps a group element* a *to* $x \circ a$.

Use standard permutation notation: Every permutation can be broken down into disjoint cycles, by following around repeated application of the permutation to one element. The permutation that maps E to F, F to G, and G to E is written (EFG) or, equivalently, (FGE) or GEF). The permutation that maps P to Q is written as (PS) or (QP). The combination of the two is written, for example, as (EFG)(PQ). An object Y that is mapped to itself may be omitted, or indicated as (Y).

ii. *Write out how these group elements act, viewing each group element as acting on the vertices A, B, and C.*
iii. *Write out how these group elements act, viewing each of them as permuting the "front" and the "back" of the object.*
iv. *Consider these motions to be transformations of the plane, and write them out as* 2×2 *matrices.*
v. *Consider these motions to be transformations of an object in 3D, in which the "reflections" are half-circle rotations around one of the mirror lines in the diagram. Write them out as* 3×3 *matrices.*
vi. *Verify that the subset* $T = \{I, R, S\}$ *is a subgroup. Write out its left and right cosets. Are they the same?*
vii. *Verify that the subset* $V_A = \{I, M_A\}$ *is a subgroup. Write out its left and right cosets. Are they the same?*

Answers

Write out a multiplication table for the group:

$x \circ a$		a					
		I	R	S	M_A	M_B	M_C
x	I	I	R	S	M_A	M_B	M_C
	R	R	S	I	M_C	M_A	M_B
	S	S	I	R	M_B	M_C	M_A
	M_A	M_A	M_B	M_C	I	R	S
	M_B	M_B	M_C	M_A	S	I	R
	M_C	M_C	M_A	M_B	R	S	I

So, the permutations are:

$\pi(I) = (I)(R)(S)(M_A)(M_B)(M_C)$ (i.e., the trivial permutation)

$\pi(R) = (IRS)(M_A M_C M_B)$

$\pi(S) = (ISR)(M_A M_B M_C)$

$\pi(M_A) = (IM_A)(RM_B)(SM_C)$

$\pi(M_B) = (IM_B)(RM_C)(SM_A)$

$\pi(M_C) = (IM_C)(RM_A)(SM_B)$

ii.

I moves $(A)(B)(C)$ (i.e., the trivial permutation)

R moves (ABC)

S moves (ACB)

M_A moves (BC)

M_B moves (AC)

M_C moves (AB)

iii. I, R, and S act trivially (i.e., the front stays the front, the back stays the back); M_A, M_B, and M_C interchange the front and the back.

iv. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, $S = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ (or their transposes, depending on whether you are rotating the object or the coordinates).

$M_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ by inspection; then, using the group multiplication table,

$$M_B = SM_A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}; M_C = RM_A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

v. We'll put the out-of-the-paper dimension as the third dimension. I, R, and S are rotations around an axis perpendicular to the paper (i.e., do not change the third dimension), so they are the above 2×2 matrices, augmented by a 1 on the diagonal for the third dimension:

$$I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ R=\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&1\end{pmatrix},\ S=\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&1\end{pmatrix}.$$

M_A is a half-circle rotation about the vertical axis (second coordinate), so $M_A=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$. Then, as in (iv),

$$M_B=SM_A=\begin{pmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&-1\end{pmatrix};M_C=RM_A=\begin{pmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&-1\end{pmatrix}.$$

vi. This can be verified by inspection of the group multiplication table, but there are many ways with more insight: From (iii), $T=\{I,R,S\}$ are all of the group elements that keep the front and the back invariant. From (i), $T=\{I,R,S\}$ are all the elements of the group that don't mix the rotations and the mirrors. Or, they are all the elements that preserve "handedness" of an image drawn on the triangle. Or, from (iv) and using determinants, and the fact that the determinant of a product is the product of the determinants -- they are all of the group elements with a determinant of 1.Or, they are all of the group elements whose permutation representations, either in (i), (ii) or (iii), have even parity.

Right cosets are T and the subset of the three mirrors. Left cosets the same.

vii. For $V_A=\{I,M_A\}$, all we need to check is that M_A^2 is in V_A; this holds since $M_A^2=I$.
Right cosets are $V_AR=\{IR,M_AR\}=\{R,M_B\}$, and $V_AS=\{IS,M_AS\}=\{S,M_C\}$.
Left cosets are $RV_A=\{RI,RM_A\}=\{R,M_C\}$, and $SV_A=\{SI,SM_A\}=\{S,M_B\}$.
Right and left cosets are not the same.

Note though that $RV_AR^{-1}=\{RIR^{-1},RM_AS\}=\{I,M_B\}$, another subgroup. Also, $SV_AS^{-1}=\{SIS^{-1},SM_AR\}=\{I,M_C\}$, also a subgroup.

The distinction between the behavior of $T=\{I,R,S\}$ and V_A will motivate the definition of a "normal" subgroup.