Groups, Fields, and Vector Spaces

Homework #2 (2024-2025), Questions

Q1: Homomorphisms, kernels, normal subgroups

We showed that for any homomorphism $\varphi : G \to H$, the kernel of φ, i.e., the elements $g \in G$ for which $\varphi(g) = e_H$, is a subgroup of G. Show that it is a normal subgroup.

Q2: Inner and outer automorphisms

A. For $G = \mathbb{Z}_n$ (the cyclic group of order n), determine all of the automorphisms.

B. Recall: For any group G, the automorphism group $A(G)$ is the group of isomorphisms of G, i.e, one-to-one mappings φ from G to G which preserve the group operation in G. The group operation in $A(G)$ is composition: $\varphi_1 \circ \varphi_2$ is the automorphism of G defined by $\varphi_1 \circ \varphi_2(g) = \varphi_1\left(\varphi_2(g)\right)$. We also said that there is a special set of automorphisms, the "inner" automorphisms. For any element α in G, the inner automorphism φ_α is defined by $\varphi_\alpha(g) = \alpha g \alpha^{-1}$. We called the mapping from G to $A(G)$ the "adjoint" map, and noted that it is a homomorphism from G to a subgroup of (and possibly all of) $A(G)$. We also noted that $Adj : G \to A(G)$ is, itself, a homomorphism: For any $g \in G$,
$\left(\varphi_\alpha \circ \varphi_\beta\right)(g) = \varphi_\alpha\left(\varphi_\beta(g)\right) = \varphi_\alpha\left(\beta g \beta^{-1}\right) = \alpha\left(\beta g \beta^{-1}\right)\alpha^{-1} = \alpha\beta g \beta^{-1}\alpha^{-1} = (\alpha\beta) g (\alpha\beta)^{-1} = \varphi_{\alpha\beta}(g)$, so
$Adj(\alpha) \circ Adj(\beta) = Adj(\alpha\beta)$.

Show that the inner automorphisms $I(G)$ are a normal subgroup of $A(G)$.

Q3: Direct sums of groups

Given two groups G and H with group operations $\circ_G$ and $\circ_H$, the direct sum $G \oplus H$ is a group consisting of ordered pairs of elements (g, h), with the group operation defined by
$(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ_G g_2, h_1 \circ_H h_2)$.

A. Convince yourself that $G \oplus H$ is a group.

B. If G and H are finite, with sizes $|G|$ and $|H|$, what is the size of $G \oplus H$?

C. Consider $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. What is its automorphism group?

Q4: A challenge

$G \oplus H \oplus K$ is defined analogously as a group of ordered triplets. What is the size of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and what is the size of its automorphism group?