Homework #1 (2024-2025), Answers

Q1. Some important transfer functions

A. *The "boxcar",which averages a signal* $s(t)$ *over a previous interval* τ *:* $f(t)=\begin{cases}\dfrac{1}{\tau}, 0\le t\le\tau\\ 0, \textit{otherwise}\end{cases}$.

Compute the transfer function, $\hat{f}(\omega)=\int\limits_{-\infty}^{\infty} e^{-i\omega t}f(t)dt$.

$$\hat{f}(\omega)=\int\limits_{-\infty}^{\infty} e^{-i\omega t}f(t)dt=\frac{1}{\tau}\int\limits_{0}^{\tau} e^{-i\omega t}dt=\frac{1}{-i\omega\tau}\left(e^{-i\omega t}\right)\Big|_0^\tau=\frac{1}{i\omega\tau}\left(1-e^{-i\omega\tau}\right).$$

B. *The delay, i.e., a filter for which the response to a signal* $s(t)$ *is* $r(t)=s(t-\tau)$, *the impulse response is* $f(t)=\delta(t-\tau)$. *Compute the transfer function,* $\hat{f}(\omega)=\int\limits_{-\infty}^{\infty} e^{-i\omega t}f(t)dt$.

$$\hat{f}(\omega)=\int\limits_{-\infty}^{\infty} e^{-i\omega t}f(t)dt=\int\limits_{0}^{\tau} e^{-i\omega t}\delta(t-\tau)dt=e^{-i\omega\tau}$$

C. *Non-causal boxcar averaging, i.e., averaging a signal* $s(t)$ *over the interval from* $-\tau/2$ *to* $+\tau/2$. *Compute the transfer function.*

This is the boxcar of part A, followed by a "delay" of $-\tau/2$ (part B) So it is the product of the corresponding transfer functions:

$$\hat{f}(\omega)=\frac{1}{i\omega\tau}\left(1-e^{-i\omega\tau}\right)e^{i\omega\tau/2}=\frac{1}{i\omega\tau}\left(e^{i\omega\tau/2}-e^{-i\omega\tau/2}\right)=\frac{2}{i\omega\tau}\left(\frac{e^{i\omega\tau/2}-e^{-i\omega\tau/2}}{2}\right)=\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)},$$ often written as

$\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$. Note that this has a zero at $\omega\tau/2=n\pi$ (except for $n=0$), so the "boxcar average" eliminates certain frequencies.

D. *The derivative, method 1: Consider a filter* f *whose output is the time-derivative of the input. First, for any signal* $s(t)$. $s'(t)=\lim_{\tau\to 0}\frac{s(t)-s(t-\tau)}{\tau}$. *Say* f_τ *yields* $\frac{s(t)-s(t-\tau)}{\tau}$. *Using part B, determine* $\hat{f}_\tau(\omega)$ and then $\hat{f}(\omega)=\lim_{\tau\to 0}\hat{f}_\tau(\omega)$.

For a given τ, say f_τ subtracts a signal after a delay of τ and divides by τ. Then $\hat{f}_\tau(\omega)=\frac{1}{\tau}\left(1-e^{-i\omega\tau}\right)$. Since $e^a=1+a+o(a^2)$,

$$\hat{f}(\omega)=\lim_{\tau\to 0}\frac{1}{\tau}\left(1-e^{-i\omega\tau}\right)=\lim_{\tau\to 0}\frac{1}{\tau}\left(1-\left(1-i\omega\tau+o(\omega)\right)\right)$$
$$=\lim_{\tau\to 0}\frac{1}{\tau}\left(i\omega\tau\right)=i\omega$$

(Technical note, the limit is not uniform in ω.)

E. *The derivative, method 2: If* $r(t)=s'(t)$, $\hat{r}(\omega)$ *can be directly determined from* $\hat{s}(\omega)$, *by expressing* $s(t)$ *in terms of* $\hat{s}(\omega)$ *and then differentiating. Then* $\hat{r}(\omega)=\hat{f}(\omega)\hat{s}(\omega)$.

$s(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}\hat{s}(\omega)d\omega$, so

$$s'(t)=\frac{ds}{dt}=\frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty}e^{i\omega t}\hat{s}(\omega)d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}e^{i\omega t}\hat{s}(\omega)d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}i\omega e^{i\omega t}\hat{s}(\omega)d\omega\quad.$$

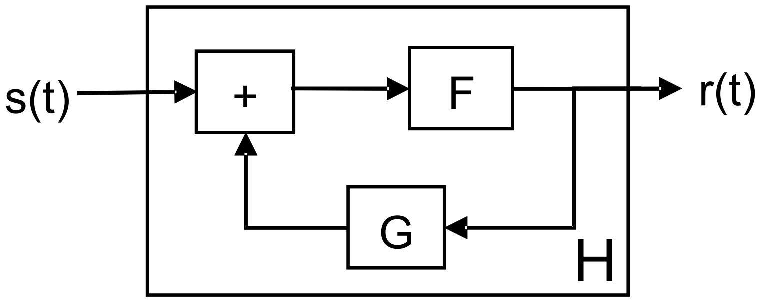
(Moving the derivative inside the integral sign requires that the integral has sufficiently nice convergence properties.) Since $r(t)=s'(t)$, this is also the Fourier representation of r. So $r(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}i\omega e^{i\omega t}\hat{s}(\omega)d\omega$ but also $r(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}\hat{r}(\omega)d\omega$. So $\hat{r}(\omega)=i\omega\hat{s}(\omega)$, and $\hat{f}(\omega)=i\omega$.

F. *From either D or E, what is the transfer function* $\hat{f}_n(\omega)$ *corresponding to the* n *th derivative?*

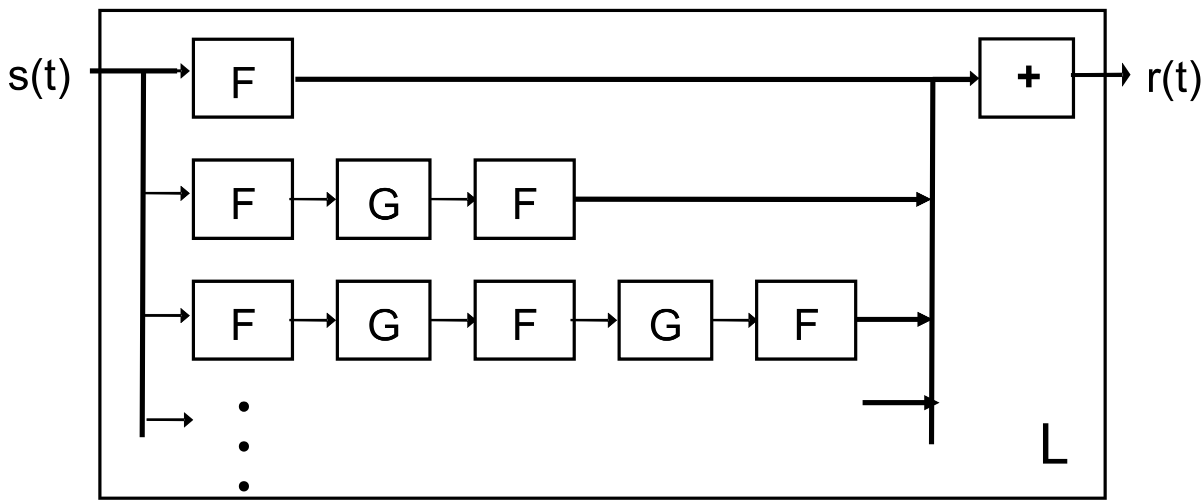
$\hat{f}_n(\omega)=\left(\hat{f}_1(\omega)\right)^n$, where $\hat{f}_1(\omega)$ is the first-derivative transfer function of part D or E. So $\hat{f}_n(\omega)=\left(\hat{f}_1(\omega)\right)^n=\left(i\omega\right)^n$.

Q2. Feedback and feedforward

We had determined the transfer function of the composite system H diagrammed here (worked out in class with the feedback signal multiplied by an arbitrary amount k; here, for simplicity, with $k=1$). For this system, $\hat{h}(\omega)=\dfrac{\hat{f}(\omega)}{1-\hat{f}(\omega)\hat{g}(\omega)}$.



Now, consider the following system, of parallel feedforward elements:



What is its transfer function, $\hat{l}(\omega)$? How does it compare to $\hat{h}(\omega)$?

L is a parallel combination of systems, each of which is a series combination.

So $\begin{aligned}\hat{l}(\omega)&=\hat{f}(\omega)+\hat{f}(\omega)g(\hat{\omega})\hat{f}(\omega)+\hat{f}(\omega)g(\hat{\omega})\hat{f}(\omega)g(\hat{\omega})\hat{f}(\omega)+...\\&=\hat{f}(\omega)+\hat{f}(\omega)g(\hat{\omega})\hat{f}(\omega)+\hat{f}(\omega)\left(g(\hat{\omega})\hat{f}(\omega)\right)^2+...\end{aligned}$, i.e., a sum of an infinite geometric series

with ratio $\hat{f}(\omega)g(\hat{\omega})$. Then $\hat{l}(\omega)=\sum_{m=0}^{\infty}\hat{f}(\omega)\left(g(\hat{\omega})\hat{f}(\omega)\right)^m=\hat{f}(\omega)\sum_{m=0}^{\infty}\left(g(\hat{\omega})\hat{f}(\omega)\right)^m=\dfrac{\hat{f}(\omega)}{1-g(\hat{\omega})\hat{f}(\omega)}$, which is identical to $\hat{h}(\omega)=\dfrac{\hat{f}(\omega)}{1-\hat{f}(\omega)\hat{g}(\omega)}$. Note that the feedforward geometric series converges at all frequencies for which the feedback is stable. Note also that a feedback system can always be emulated by a feedforward system.

Q3. The Fourier transform of a Gaussian

We evaluate $J(D,u)=\int\limits_{-\infty}^{\infty} e^{-\omega^2 D/2} e^{i\omega u} d\omega$.

A. *First, consider* $I(V)=\int\limits_{-\infty}^{\infty} e^{-x^2/2V} dx$*, the integral of a non-normalized Gaussian. Note that* I^2 *can be considered a two-dimensional integral (say, in* x *and* y*), and also an integral in polar coordinates with* $r^2 = x^2 + y^2$*. In polar coordinates, the integral is straightforward. This yields* I^2 *and hence* I*.*

$$I^2 = \int\limits_{-\infty}^{\infty} e^{-x^2/2V} dx \int\limits_{-\infty}^{\infty} e^{-y^2/2V} dy = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-x^2/2V} e^{-y^2/2V} dxdy = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-(x^2+y^2)/2V} dxdy .$$

Changing to polar coordinates, with $dxdy = rdrd\theta$:

$I^2 = \int\limits_{0}^{\infty}\int\limits_{0}^{2\pi} e^{-r^2/2V} rdrd\theta = 2\pi \int\limits_{0}^{\infty} e^{-r^2/2V} rdr$. With $t = r^2/2$, $dt = rdr$, and

$I^2 = 2\pi \int\limits_{0}^{\infty} e^{-t/V} dt = -2\pi V \left(e^{-t/V}\right)\Big|_0^{\infty} = 2\pi V$. So $I = \sqrt{2\pi V}$.

B. *Evaluate* $\int\limits_{-\infty}^{\infty} e^{-\omega^2 D/2} e^{i\omega u} d\omega$ *by completing the square in the exponent.*

Focusing on the exponent,

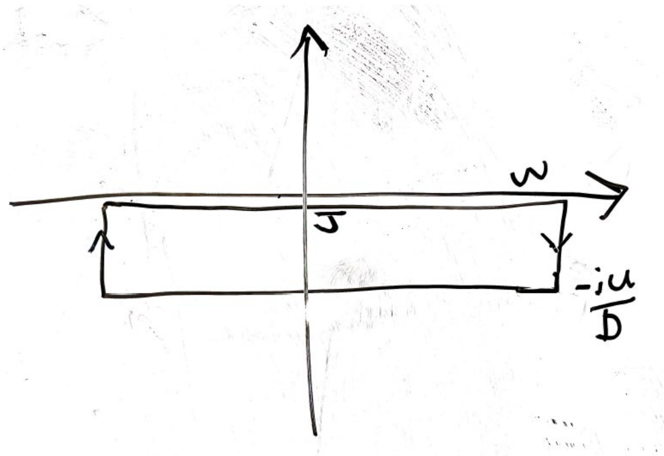
$$\frac{\omega^2 D}{2} - i\omega u = \frac{D}{2}\left(\omega^2 - \frac{2iu}{D}\omega\right) = \frac{D}{2}\left(\omega^2 - \frac{2iu}{D}\omega - \frac{u^2}{D^2}\right) + \frac{D}{2}\left(\frac{u^2}{D^2}\right)$$
$$= \frac{D}{2}\left(\omega - \frac{iu}{D}\right)^2 + \frac{D}{2}\left(\frac{u^2}{D^2}\right)$$

$$J(D,u) = \int\limits_{-\infty}^{\infty} e^{-\omega^2 D/2} e^{i\omega u} d\omega = \int\limits_{-\infty}^{\infty} \exp\left(-\left[\frac{D}{2}\left(\omega - \frac{iu}{D}\right)^2 + \frac{D}{2}\left(\frac{u^2}{D^2}\right)\right]\right) d\omega$$

So $= \exp\left(-\frac{D}{2}\left(\frac{u^2}{D^2}\right)\right) \int\limits_{-\infty}^{\infty} \exp\left(-\frac{D}{2}\left(\omega - \frac{iu}{D}\right)^2\right) d\omega$.

$$= \exp\left(-\frac{u^2}{2D}\right) \int\limits_{-\infty}^{\infty} \exp\left(-\frac{D}{2}\left(\omega - \frac{iu}{D}\right)^2\right) d\omega$$

For the last integral: Consider shifting the integration away from the positive real axis by $-iu/D$. This integral and the original integral must be the same: together, they form a closed contour by adding vertical components at infinity. These components contribute nothing, and the contour contains no poles. So the contour integral is zero, and the integral over the real axis, or the displaced parallel line, are the same.



So, take $v = \omega - iu / D$.

$$J(D,u) = \exp\left(-\frac{u^2}{2D}\right)\int_{-\infty}^{\infty} \exp\left(-\frac{D}{2}v^2\right)dv = \exp\left(-\frac{u^2}{2D}\right)I(1/D) = \sqrt{\frac{2\pi}{D}}\exp\left(-\frac{u^2}{2D}\right)$$