Linear Systems: Black Boxes and Beyond

Homework #2 (2024-2025), Answers

Q1. Multi-input, multi-output systems and coherence

Consider a linear system L with m inputs and n outputs. It can be characterized by an array of impulse responses, $L_{mn}(\tau)$, which specify the response of the nth output to an impulse on the mth input, or, equivalently, an array of transfer functions $\hat{L}_{mn}(\omega)=\int_0^\infty e^{-i\omega t}L_{mn}(t)dt$ that specify the sinusoidal component at ω on the nth output produced by a unit sinusoid on the mth input. We can also denote the array of transfer functions by $\hat{L}(\omega)$.

A. *Given two such systems in series, say A with m inputs and n outputs and transfer functions $\hat{A}_{nm}(\omega)$, and B, which takes these n outputs as its input and produces p outputs, with transfer functions $\hat{B}_{pn}(\omega)$, what are the transfer functions $\tilde{L}_{pm}(\omega)$ of the composite system consisting of A followed by B?*

Let $\hat{x}_m(\omega)$ be the inputs to A, $\hat{y}_n(\omega)$ the outputs of A *and the inputs to* B, and $\hat{z}_p(\omega)$ *the outputs of* B.

Then $\hat{y}_n(\omega)=\sum_m \hat{A}_{nm}(\omega)\hat{x}_m$, and $\hat{z}_p(\omega)=\sum_n \hat{B}_{pn}(\omega)\hat{y}_n$, so

$$\begin{aligned}\hat{z}_p(\omega)&=\sum_n \hat{B}_{pn}(\omega)\left(\sum_m \hat{A}_{nm}(\omega)\hat{x}_m\right)=\sum_{m,n}\hat{B}_{pn}(\omega)\hat{A}_{nm}(\omega)\hat{x}_m\\&=\sum_m\left(\sum_n \hat{B}_{pn}(\omega)\hat{A}_{nm}(\omega)\right)\hat{x}_m\end{aligned}$$

. Then the input-output relationship of the composite system is given by $\hat{z}_p(\omega)=\sum_m \hat{L}_{pm}(\omega)\hat{x}_m$, i.e., $\hat{L}_{pm}(\omega)=\sum_n \hat{B}_{pn}(\omega)\hat{A}_{nm}(\omega)$. More compactly, we could have simply regarded the parallel signals as a column vector, and then $\hat{y}(\omega)=\hat{A}(\omega)\hat{x}(\omega)$, $\hat{z}(\omega)=\hat{B}(\omega)\hat{y}(\omega)$, and $\hat{z}(\omega)=\hat{B}(\omega)\hat{y}(\omega)=\hat{B}(\omega)\hat{A}(\omega)\hat{x}(\omega)$, so $\hat{L}(\omega)=\hat{B}(\omega)\hat{A}(\omega)$.

B. *For systems with the same number of inputs and outputs (i.e., $m=n=p$ above), does the order of composition matter?*

Yes, since matrix multiplication is not commutative.

C. *With A as above (m inputs, n outputs, transfer functions $\hat{A}_{nm}(\omega)$): How does the cross-spectral matrix of the output $P_{Y_j,Y_k}(\omega)$ relate to the cross-spectral matrix of the input, $P_{X_j,X_k}(\omega)$? What if the inputs consist of independent Gaussian noises with unit spectral density?*

   An element of the cross-spectral matrix is given by $P_{Y_j,Y_k}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle F(y_j,\omega,T,T_0)\overline{F(y_k,\omega,T,T_0)}\right\rangle$, where (as in the notes) F is a Fourier estimate for an interval T beginning at T_0. A Fourier estimate for the output y_j is approximated by $F(y_j,\omega,T,T_0)\approx\sum_m A_{jm}(\omega)F(x_m,\omega,T,T_0)$, with the approximation becoming exact as $T \to \infty$. So

$$P_{Y_j,Y_k}(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle \left(\sum_m A_{jm}(\omega)F(x_m,\omega,T,T_0)\right)\overline{\left(\sum_n A_{jk}(\omega)F(x_n,\omega,T,T_0)\right)}\right\rangle$$
$$= \sum_{m,n} A_{jm}(\omega)\overline{A_{jk}(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle F(x_m,\omega,T,T_0)\overline{F(x_n,\omega,T,T_0)}\right\rangle \quad ,$$
$$= \sum_{m,n} A_{jm}(\omega)\overline{A_{k,n}(\omega)}P_{X_j,X_k}(\omega)$$

since the last limit is the definition of the cross-spectrum between two inputs.

Denoting the cross-spectral matrix of the input by P_X and of the output by P_Y:

$$P_Y(\omega) = \hat{A}(\omega)P_X(\omega)\hat{A}(\omega)^*$$

If the input is independent, unit-power Gaussians, P_X is the $m \times m$ identity and $P_Y(\omega) = \hat{A}(\omega)\hat{A}(\omega)^*$.

D. *Consider two m-input, m-output systems, A and B. For what conditions on A are the cross-spectral matrices of B, and of L, consisting of A followed by B, identical?*

For B alone, the cross-spectral matrix is $P_B(\omega) = \hat{B}(\omega)\hat{B}(\omega)^*$. For A followed by B, the composite transfer function is $\hat{L}(\omega) = \hat{B}(\omega)\hat{A}(\omega)$, so $P_L(\omega) = \hat{L}(\omega)\hat{L}(\omega)^* = \hat{B}(\omega)\hat{A}(\omega)\hat{A}(\omega)^*\hat{B}(\omega)^*$. These are guaranteed identical if $\hat{A}(\omega)\hat{A}(\omega)^* = I$, i.e., that $\hat{A}(\omega)$ is unitary (for all ω).

Q2. Hermite polynomials and generating functions

Hermite polynomials – orthogonal polynomials with respect to a Gaussian -- play a major role in extending input-output analysis to nonlinear systems. This is because of both the Central Limit Theorem and Price's Theorem (see Question 3). Question 4 illustrates this extension, and can be done without first doing Q2 and Q3.

First, we establish the orthogonality of Hermite polynomials and then prove Price's Theorem using generating functions. If you haven't seen generating functions, they are a good thing to have in your toolkit. Demonstrating orthogonality of the Hermite polynomials is the "warm-up exercise."

In our standardization, the m th Hermite polynomial $h_m(x)$ is defined as the coefficient of t^m in $\exp(xt - \frac{t^2}{2})$, specifically, $\sum_{m=0}^{\infty} h_m(x)\frac{t^m}{m!} = \exp(xt - \frac{t^2}{2})$. In this standardization, $h_m(x)$ has a leading coefficient 1.

Show that the Hermite polynomials are orthogonal with respect to a unit Gaussian, namely, that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_m(x)h_m(x)e^{-x^2/2}dx = \begin{cases} 0, & m \neq n \\ m!, & m = n \end{cases}.$$

A. *With* $I_{m,n}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h_m(x)h_n(x)e^{-x^2/2}dx$, *express* $I(s,t)=\sum_{m,n=0}^{\infty}I_{m,n}\frac{s^m}{m!}\frac{t^n}{n!}$ *as an integral of an exponential, using the generating-function definition of the Hermites.*

$$I(s,t)=\frac{1}{\sqrt{2\pi}}\sum_{m,n=0}^{\infty}\frac{s^m}{m!}\frac{t^n}{n!}\int_{-\infty}^{\infty}h_m(x)h_n(x)e^{-x^2/2}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{m,n=0}^{\infty}h_m(x)\frac{s^m}{m!}h_n(x)\frac{t^n}{n!}e^{-x^2/2}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(xs-\frac{s^2}{2})\exp(xt-\frac{t^2}{2})e^{-x^2/2}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}(x^2-2xs-2xt+s^2+t^2)\right)dx$$

B. *Integrate* $I(s,t)$ *(complete the square in the exponent and use* $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2/2}dx=1$*).*

Focusing on the exponent: $x^2-2xs-2xt+s^2+t^2=\left(x-(s+t)\right)^2-2st$.

So,

$$I(s,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}(x^2-2xs-2xt+s^2+t^2)\right)dx$$

$$=\exp(st)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\left(x-(s+t)\right)^2\right)dx$$

After a change of variables $z=x-(s+t)$, $I(s,t)=\exp(st)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}z^2\right)dx=\exp(st)$.

C. Equate the expressions in A and B for $I(s,t)$ term-by-term to determine $I_{m,n}$.

$\sum_{m,n=0}^{\infty}I_{m,n}\frac{s^m}{m!}\frac{t^n}{n!}=\exp(st)$, but $\exp(st)=\sum_{m=0}^{\infty}\frac{(st)^m}{m!}$. So $\sum_{m,n=0}^{\infty}I_{m,n}\frac{s^m}{m!}\frac{t^n}{n!}=\sum_{m=0}^{\infty}\frac{s^mt^m}{m!}$.

Equating term by term shows that for $m\neq n$, $I_{m,n}=0$ (since there's no corresponding term on the right). And for $m=n$, $\frac{I_{m,m}}{m!m!}=\frac{1}{m!}$, so $I_{m,m}=m!$.

Q3. Price's Theorem

Price's Theorem states that if two variables are drawn from a correlated Gaussians (say, x *and* y*, each with zero mean and unit variance, and correlation* $\rho=\langle xy\rangle$*), then, for any Hermite polynomials* h_m *and* h_n *(defined below),* $\langle h_m(x)h_n(y)\rangle=0$ *if* $m\neq n$*, and* $\langle h_m(x)h_m(y)\rangle=m!\rho^m$*. This is crucial to extending the cross-correlation approach to nonlinear systems.*

First, we set up correlated unit-variance, zero-mean Gaussian variables. Let u *and* v *be UNcorrelated unit-mean Gaussian variables, and* $x=u\cos\theta+v\sin\theta$*,* $y=u\sin\theta+v\cos\theta$ *(note, not a rotation).*

A. *Determine* $\langle x^2\rangle$*,* $\langle y^2\rangle$*, and* $\rho=\langle xy\rangle$

$$\left\langle x^2\right\rangle = \left\langle (u\cos\theta + v\sin\theta)^2\right\rangle = \left\langle u^2\cos^2\theta + v^2\sin^2\theta + 2uv\cos\theta\sin\theta\right\rangle$$

Using $\left\langle u^2\right\rangle = \left\langle v^2\right\rangle = 1$ and $\left\langle uv\right\rangle = 0$:

$$= \left\langle u^2\cos^2\theta\right\rangle + \left\langle v^2\sin^2\theta\right\rangle + \left\langle 2uv\cos\theta\sin\theta\right\rangle$$
$$= \left\langle u^2\right\rangle\cos^2\theta + \left\langle v^2\right\rangle\sin^2\theta + 2\left\langle uv\right\rangle\cos\theta\sin\theta$$
$$= \cos^2\theta + \sin^2\theta = 1$$

and similarly for $\left\langle y^2\right\rangle = 0$.

But
$$\left\langle x^2\right\rangle = \left\langle (u\cos\theta + v\sin\theta)(u\sin\theta + v\cos\theta)\right\rangle = \left\langle u^2\cos\theta\sin\theta + v^2\cos\theta\sin\theta + uv(\cos^2\theta + \sin^2\theta)\right\rangle$$
$$= \left\langle u^2\right\rangle\cos\theta\sin\theta + \left\langle v^2\right\rangle\cos\theta\sin\theta + \left\langle uv\right\rangle(\cos^2\theta + \sin^2\theta) = 2\cos\theta\sin\theta = \sin 2\theta .$$

So $\rho = \left\langle xy\right\rangle = \sin 2\theta$.

B. *We want to calculate $J_{m,n} = \left\langle h_m(x)h_n(y)\right\rangle$. Rather than integrate over a pair of correlated Gaussians in x and y,, we use the underlying uncorrelated Gaussians u and v. So*

$$J_{m,n} = \left\langle h_m(x)h_n(y)\right\rangle = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_m(u\cos\theta + v\sin\theta)h_n(u\sin\theta + v\cos\theta)e^{-u^2/2}e^{-v^2/2}dudv .$$

Write the generating function $J(s,t) = \sum_{m,n=0}^{\infty} J_{m,n}\frac{s^m}{m!}\frac{t^n}{n!}$ using the generating function for the Hermites, integrate, and equate term-by-term to demonstrate the claim of Price's Theorem.

$$J(s,t) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \sum_{m,n=0}^{\infty}\frac{s^m}{m!}\frac{t^n}{n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_m(u\cos\theta + v\sin\theta)h_n(u\sin\theta + v\cos\theta)e^{-u^2/2}e^{-v^2/2}dudv$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left((u\cos\theta + v\sin\theta)s - \frac{s^2}{2} + (u\sin\theta + v\cos\theta)t - \frac{t^2}{2} - \frac{u^2}{2} - \frac{v^2}{2}\right)dudv .$$

Note that the integral separates, so

$$J(s,t) = \exp(-\frac{s^2}{2} - \frac{t^2}{2})\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(us\cos\theta + ut\sin\theta - \frac{u^2}{2}\right)du\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(vs\sin\theta + vt\cos\theta - \frac{v^2}{2}\right)dv\right).$$

Each factor can be handled by completing the square:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(us\cos\theta + ut\sin\theta - \frac{u^2}{2}\right)du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}(2us\cos\theta + 2ut\sin\theta - u^2)\right)du$$
$$= \exp\left(\frac{1}{2}(s\cos\theta + t\sin\theta)^2\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}(u - s\cos\theta - t\sin\theta)^2\right)du$$

, and similarly

$$= \exp\left(\frac{1}{2}(s\cos\theta + t\sin\theta)^2\right)$$

for the second factor,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(vs\sin\theta+vt\cos\theta-\frac{v^2}{2}\right)dv=\exp\left(\frac{1}{2}(s\sin\theta+t\cos\theta)^2\right).$$

So

$$J(s,t)=\exp(-\frac{s^2}{2}-\frac{t^2}{2})\exp\left(\frac{1}{2}(s\cos\theta+t\sin\theta)^2\right)\exp\left(\frac{1}{2}(s\sin\theta+t\cos\theta)^2\right)$$

$$=\exp\left(\frac{1}{2}\left(-s^2-t^2+s^2\cos^2\theta+t^2\cos^2\theta+2st\cos\theta\sin\theta+s^2\sin^2\theta+t^2\cos^2\theta+2st\sin\theta\cos\theta\right)\right).$$

$$=\exp\left(2st\cos\theta\sin\theta\right)=\exp(st\sin 2\theta)=\exp(st\rho)$$

Going back to the original definition of $J(s,t)$:

$$J(s,t)=\sum_{m,n=0}^{\infty}\langle h_m(x)h_n(y)\rangle\frac{s^m}{m!}\frac{t^n}{n!}=\exp(st\rho)=\sum_{m}^{\infty}\rho^m\frac{s^m t^m}{m!}.$$

Equating term-by-term: $\langle h_m(x)h_n(y)\rangle$ is zero if $m\neq n$, and $\frac{\langle h_m(x)h_m(y)\rangle}{m!m!}=\frac{\rho^m}{m!}$, i.e., $\langle h_m(x)h_m(y)\rangle=m!\rho^m$.

Q4. Application to input-output analysis of nonlinear systems.

Consider a composite system consisting of a linear filter followed by a static nonlinear system. Specifically, the linear system L has an impulse-response $L(\tau)$, that produces a response $q(t)$ to an input $s(t)$, and is followed by a static nonlinear system N whose response to q is given by a nonlinear function $r=N(q)$. That is, the response of N at any given time depends only on its input at that time, and not on previous values of the input.

We analyze the response of this composite system when its input is a Gaussian noise of unit variance.

A. *Let V be the variance of $q(t)$ when the input $s(t)$ is a unit-variance Gaussian. Provided that $\int_{-\infty}^{\infty}\left(N(q)\right)^2 e^{-q^2/2V}dq$ is finite, $N(q)$ can be expanded in terms of Hermite polynomials as $N(q)=\sum_{k=0}^{\infty}\alpha_k h_k\left(\frac{q}{\sqrt{V}}\right)$. Using the orthogonality of the Hermite polynomials (Q2), namely, that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h_m(x)h_m(x)e^{-x^2/2}dx=\begin{cases}0,\ m\neq n\\ m!,\ m=n\end{cases}$, determine α_k. Hint: Consider $N(q)$ as a vector in a Hilbert space with inner product $(f,g)_H=\frac{1}{\sqrt{2\pi V}}\int_{-\infty}^{\infty}f(q)\overline{g(q)}e^{-q^2/2V}dq$. Then think of projecting it onto the one-dimensional subspace spanned by $h_m\left(\frac{q}{\sqrt{V}}\right)$, using that inner product. In other words, the α_k aree the coordinates of $N(q)$ in the basis set consisting of the $h_m\left(\frac{q}{\sqrt{V}}\right)$.*

The vectors $h_m\left(\frac{q}{\sqrt{V}}\right)$ are orthogonal in the inner product $(\ ,\ )_H$, as this is a change of variables from the unit Hermites:

$$(h_m\left(\frac{q}{\sqrt{V}}\right), h_n\left(\frac{q}{\sqrt{V}}\right))_H = \frac{1}{\sqrt{2\pi V}}\int_{-\infty}^{\infty} h_m\left(\frac{q}{\sqrt{V}}\right)h_n\left(\frac{q}{\sqrt{V}}\right)e^{-q^2/2V}dq = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_m(x)h_n(x)e^{-x^2/2}dx = m!\delta_{m,n}$$

So, $(N(q), h_m\left(\frac{q}{\sqrt{V}}\right))_H = \sum_{k=0}^{\infty}\alpha_k (h_k\left(\frac{q}{\sqrt{V}}\right), h_m\left(\frac{q}{\sqrt{V}}\right))_H = \sum_{k=0}^{\infty}\alpha_k m!\delta_{k,m} = m!\alpha_m$.

Then $\alpha_m = \frac{1}{m!}(N(q), h_m\left(\frac{q}{\sqrt{V}}\right))_H = \frac{1}{\sqrt{2\pi V}}\int_{=\infty}^{\infty} N(q)h_m\left(\frac{q}{\sqrt{V}}\right)e^{-q^2/2V}dq$. *Or, more compactly,*

$$\alpha_m = \frac{(N(q), h_m\left(\frac{q}{\sqrt{V}}\right))_H}{(h_m\left(\frac{q}{\sqrt{V}}\right), h_m\left(\frac{q}{\sqrt{V}}\right))_H}.$$

B. *Using Price's theorem (Q3), determine the cross-correlation between a Hermite polynomial function of the stimulus and system's response, i.e.,* $Z_n(\tau) = \langle r(t)h_n(s(t-\tau))\rangle$.

The response is given by $r(t) = N(q(t)) = \sum_{k=0}^{\infty}\alpha_k h_k\left(\frac{q(t)}{\sqrt{V}}\right)$, where the α_k are determined as in part A.

$$Z_n(\tau) = \langle r(t)h_n(s(t-\tau))\rangle = \langle N(q(t))h_n(s(t-\tau))\rangle = \left\langle \sum_{k=0}^{\infty}\alpha_k h_k\left(\frac{q(t)}{\sqrt{V}}\right)h_n(s(t-\tau))\right\rangle = \sum_{k=0}^{\infty}\alpha_k\left\langle h_k\left(\frac{q(t)}{\sqrt{V}}\right)h_n(s(t-\tau))\right\rangle$$

.

$\frac{q(t)}{\sqrt{V}}$ and $s(t-\tau)$ are both Gaussians of unit variance. So Price's Theorem applies:

$\left\langle h_k\left(\frac{q(t)}{\sqrt{V}}\right)h_n(s(t-\tau))\right\rangle = n!\left\langle\left(\frac{q(t)}{\sqrt{V}}\right)(s(t-\tau))\right\rangle^n \delta_{k,n}$, and

$$Z_n(\tau) = \alpha_n n!\left\langle\left(\frac{q(t)}{\sqrt{V}}\right)s(t-\tau)\right\rangle^n = \frac{\alpha_n n!}{V^{n/2}}\langle q(t)s(t-\tau)\rangle^n.$$

Note that $\langle q(t)s(t-\tau)\rangle$ is the cross-correlation between the input and the output of the linear component. For unit-variance white noise input, this is equal to the impulse response of the linear component, since it is the Fourier transform of the cross-covariance.

This yields a diagnostic test for whether a nonlinear system is equivalent to a linear filter followed by a static nonlinearity: the cross-correlations Z_n must be proportional to point-by-point powers of the basic cross-correlation $\langle q(t)s(t-\tau)\rangle$. If this holds (but typically, it is only tested for $n = 2$), the proportionalities characterize the static nonlinearity.