Linear Transformations and Group Representations

Homework #1 (2024-2025), Questions

Q1: Characteristic equations, eigenvalues, eigenvectors

For each of the following: write the characteristic equation, find the eigenvalues, and find the eigenvectors. Determine if the operator is "normal" (i.e., commutes with its adjoint).

A. $A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$.
B. $B=\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$.
C. $C=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$.
D. $D=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

Q2: Tensor Products and Traces (similar to LTGR2223aHW, Q1)

Given a linear transformation A on a vector space V of dimension n, and a complete set of eigenvectors v_i and corresponding eigenvalues λ_i:

A. What are the eigenvectors and eigenvalues of $A\otimes A$?
B. What are the eigenvectors and eigenvalues of $sym(A\otimes A)$, the restriction of $A\otimes A$ to the symmetric part of $V\otimes V$?
C. What are the eigenvectors and eigenvalues of $anti(A\otimes A)$, the restriction of $A\otimes A$ to the antisymmetric part of $V\otimes V$?
D. What is $tr\left(A\otimes A\right)$, $tr\left(sym(A\otimes A)\right)$, and $tr\left(anti(A\otimes A)\right)$, in terms of $tr(A)$ and $tr(A^2)$?

Q3: Projections (similar to LTGR2223bHW, Q1)

Given projections P and Q on a vector space V:

A. Show that if P and Q commute, that PQ is also a projection. What is a geometric interpretation?

B. Show that if P and Q are projections but do not commute, then PQ is not a projection.

C. If P and Q commute, is $P+Q$ a projection? If not, give a condition on P and Q that guarantees that it is a projection. What is a geometric interpretation?

D. If P and Q commute, is $P+Q-PQ$ a projection? What is a geometric interpretation?

Q4: Inner products in a tensor-product space

Here we show how inner products on a pair of vector spaces can be extended to their tensor product, filling some gaps in the notes. Say the v are vectors in a Hilbert space V with inner product $\langle v, v' \rangle_V$ and similarly the w are vectors in a Hilbert space W with inner product $\langle w, w' \rangle_W$.

A. Give a natural definition for an inner product $\langle \ , \ \rangle_{V \otimes W}$ on vectors in $V \otimes W$. Show self-consistency.

B. Show that the properties of an inner product (linearity, conjugate symmetry, and positive-definiteness) hold.

C. What is the adjoint of $A \otimes B$?

D. Now that we know how to define adjoints: Given P a projection in V and Q a projection in W, is $P \otimes Q$ a projection in $V \otimes W$?

Q5: The dihedral group D_n and some of its representations

The dihedral group D_n consists of the rotations and reflections of a regular n-gon. This group is generated by a rotation R of $\frac{2\pi}{n}$ and by a mirror M. The other mirror reflections are $R^a M$ ($a = 1, ..., n-1$), and the identity. The group properties can all be derived from the relationships $R^n = M^2 = I$ (i.e., R is of order n and M is of order 2), and $MR = R^{n-1}M$ (a rotation followed by a mirror is the same as a mirror followed by a rotation in the opposite direction), without regard to a geometrical interpretation for R and M. It is a bit fussy -- even and odd values of n behave differently -- , but it is also a chance to work with groups via the abstract relationships between their generators (here, R and M) – and to appreciate how useful it is to have a geometric interpretation.

A. Determine whether all mirror reflections are in the same conjugate class as M. Since the group elements are I, R^a ($a = 1, ..., n-1$), and $R^a M$ ($a = 1, ..., n-1$), it suffices to determine gMg^{-1} for each of these (other than the identity).

B. Determine the conjugate classes of the rotations.

C. Write out the conjugate classes for D_n.

D. For definiteness, say that the n-gon has **one vertex pointing up**, and M is a reflection across the vertical axis. Consider elements of D_n as motions in the plane, and the corresponding 2-dimensional representation, say L, What is $\chi_L(R)$? What is $\chi_L(M)$?Can you construct other representations in a similar way?

E. Consider elements of D_n as permutations on the n edges and the corresponding n-dimensional representation, say E. What is $\chi_E(R)$? What is $\chi_E(M)$?

F. As in E, but consider D_n as permutations on the n vertices.

G. There is a one-dimensional representation U that maps each $g \in D_n$ to the parity of the permutation on the edges corresponding to g. What is $\chi_U(R)$? What is $\chi_U(M)$?