Linear Transformations and Group Representations

Homework #2 (2024-2025), Answers

*Q1: Orthogonality of the characters of SO(3)*

*Setup: As detailed in class notes, the group of rotations of a sphere in 3-space, $SO(3)$, has irreducible representations $L_m$, of dimension $2m+1$, for each $m \in \{0,1,2,...\}$. In $L_m$. a rotation by $\theta$ about the "$z$" axis is given by* $\begin{pmatrix} 1 & & & & & & \\ & \cos\theta & \sin\theta & & & & \\ & -\sin\theta & \cos\theta & & & & \\ & & & \ddots & & & \\ & & & & \cos m\theta & \sin m\theta \\ & & & & -\sin m\theta & \cos m\theta \end{pmatrix}$*, which, after a coordinate change, is* $\begin{pmatrix} 1 & & & & & \\ & e^{i\theta} & & & & \\ & & e^{-i\theta} & & & \\ & & & \ddots & & \\ & & & & e^{im\theta} & \\ & & & & & e^{-im\theta} \end{pmatrix}$*. So the character for the conjugate class of rotations by an angle $\theta \in [0,\pi]$ is given by* $\chi_{L_m}(R_\theta) = \sum_{k=-m}^{m} e^{ik\theta}$.

*We further stated that these characters are orthonormal, if they are properly weighted by the "mass " of their conjugate classes:* $\int_0^\pi \chi_{L_m}(R_\theta)\overline{\chi_{L_n}(R_\theta)}w(\theta)d\theta = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$*, where $w(\theta)$ is the weighting of the conjugate class $\theta$, $w(\theta) = \frac{1}{\pi}(1-\cos\theta)$. Here we demonstrate this orthonormality. Several steps but each should be straightforward.*

A. *For any real-valued function $f(\theta)$ that can be written as $f(\theta) = \sum_{k=-m}^{m} f_k e^{ik\theta}$, determine $\int_0^\pi f(\theta)d\theta$ in terms of the $f_k$.*

   For $k \neq 0$, $\int_0^\pi e^{ik\theta}d\theta = \frac{1}{ik}e^{ik\theta}\Big|_0^\pi = \frac{1}{ik}\left((-1)^k - 1\right) = \begin{cases} \frac{-2}{ik}, & k \ \ odd \\ 0, & k \ even \end{cases}$. But if $f(\theta)$ is real, i.e., $f(\theta) = \overline{f(\theta)}$, then $\sum_{k=-m}^{m} f_k e^{ik\theta} = \sum_{k=-m}^{m} f_k e^{-ik\theta}$ and therefore $f_k = f_{-k}$. So, the contribution of the terms $f_k$ and $f_{-k}$ sum to zero.
   For $k = 0$, $\int_0^\pi e^{ik\theta}d\theta = \int_0^\pi d\theta = \pi$. So $\int_0^\pi f(\theta)d\theta = \pi f_0$.

B. *Given any two functions that are of the form specified in part A (say,* $f(\theta)=\sum_{k=-m}^{m} f_k e^{ik\theta}$ *and* $g(\theta)=\sum_{l=-n}^{n} g_l e^{il\theta}$ *), express the results of multiplying them as a third such function.*

$f(\theta)g(\theta)=\sum_{k=-m}^{m} f_k e^{ik\theta}\sum_{l=-n}^{n} g_l e^{il\theta}=\sum_{k,l} f_k g_l e^{i(k+l)\theta}=\sum_{r=-(m+n)}^{m+n} e^{ir\theta}\sum_{s=\max(-m,r-n)}^{s=\min(m,r+n)} f_s g_{r-s}$, which is of the form required by A. And clearly $f(\theta)g(\theta)$ is real.

C. *Write* $w(\theta)=\frac{1}{\pi}\left(1-\cos\theta\right)$ *in the form specified in part A.*

$w(\theta)=\frac{1}{\pi}\left(1-\cos\theta\right)=\frac{1}{\pi}\left(1-\frac{e^{i\theta}+e^{-i\theta}}{2}\right)$, *so* $w(\theta)=\sum_{k=-1}^{1} w_k e^{ik\theta}$, *where* $w_0=\frac{1}{\pi}$ *and* $w_{-1}=w_{+1}=-\frac{1}{2\pi}$.

$w$ is also real. So it has the form required in A.

D. *For any non-constant real-valued function* $f(\theta)$ *of the form specified in part A, compute* $\int_0^{\pi} f(\theta)w(\theta)d\theta$ *in terms of the* $f_k$.

From parts B and C, we can write $h(\theta)=f(\theta)w(\theta)$ in the form of Part A. From part A, we know that $\int_0^{\pi} f(\theta)w(\theta)d\theta=\int_0^{\pi} h(\theta)d\theta=\pi h_0$. So we only need to calculate the $h_0$-term of $h(\theta)=f(\theta)w(\theta)$.

From B, with $n=1$, $f(\theta)w(\theta)=\sum_{r=-(m+1)}^{m+1} e^{ir\theta}\sum_{s=\max(-m,r-1)}^{s=\min(m,r+1)} f_s w_{r-s}$. We only need the term for $r=0$:

$h_0=\sum_{s=\max(-m,-1)}^{s=\min(m,1)} f_s w_{-s}$. Since $f$ is non-constant, $m\geq 1$, and then

$h_0=\sum_{s=-1}^{1} f_s w_{-s}=\frac{1}{\pi}\left(-\frac{1}{2}f_{-1}+f_0-\frac{1}{2}f_1\right)$, so $\int_0^{\pi} f(\theta)w(\theta)d\theta=\pi h_0=-\frac{1}{2}f_{-1}+f_0-\frac{1}{2}f_1$

E. *For* $f(\theta)=\chi_{L_m}(R_\theta)$, $g(\theta)=\chi_{L_n}(R_\theta)$, *and* $z(\theta)=f(\theta)g(\theta)$, *determine* $z_k$ *so that* $f(\theta)g(\theta)=\sum_{k=-(m+n)}^{m+n} z_k e^{ik\theta}$ *and, with the results of part D, demonstrate orthonormality.*

For $f(\theta)=\chi_{L_m}(R_\theta)$, $f_k=1$ for $|k|\leq m$ and 0 otherwise.
For $g(\theta)=\chi_{L_n}(R_\theta)$, $g_k=1$ for $|k|\leq n$ and 0 otherwise.

$z(\theta)=f(\theta)g(\theta)=\sum_{r=-(m+n)}^{m+n} e^{ir\theta}\sum_{s=\max(-m,r-n)}^{s=\min(m,r+n)} f_s g_{r-s}=\sum_{r=-(m+n)}^{m+n} e^{ir\theta}\sum_{s=\max(-m,r-n)}^{s=\min(m,r+n)} 1$

So $z_r$ is the number of integers $s$ over which the inner sum ranges. This range is from $\max(-m, r-n)$ to $\min(m, r+n)$ so $z_r = 1 + \min(m, r+n) - \max(-m, r-n) = 1 + \min(m, r+n) + \min(m, n-r)$ (since $\max(u, v) = -\min(-u, -v)$). Also, since $\min(u, v) = \frac{1}{2}\left(u + v - |u - v|\right)$,

$$
\begin{aligned}
z_r &= 1 + \frac{1}{2}\left(m + r + n - |m - r - n|\right) + \frac{1}{2}\left(m - r + n - |m + r - n|\right) \\
&= 1 + m + n - \frac{1}{2}\left(|m - r - n| + |m + r - n|\right)
\end{aligned}
.
$$

(This slightly messy algebra can also be viewed geometrically: each $f_k$ and $g_k$ are "plateaus", and $z_k$ is the convolution of them, so $z_k$ is a trapezoid.)

What does this look like? We only need to consider $r \geq 0$, since (as observed in part A) $z$ being real requires $z_r = z_{-r}$. For $m = n$, $z_r = 1 + 2m - \frac{1}{2}\left(|-r| + |r|\right) = 1 + 2m - |r|$. So $z_{-1} = z_{+1} = z_0 - 1$ (the short side of the trapezoid reduces to a point, so the trapezoid becomes an isosceles triangle). From D,

$$\int_0^{\pi} z(\theta) w(\theta) d\theta = -\frac{1}{2} z_{-1} + z_0 - \frac{1}{2} z_1 = -\frac{1}{2}(z_0 - 1) + z_0 - \frac{1}{2}(z_0 - 1) = 1.$$

For $m > n$ ($m < n$ just switches the roles of $m$ and $n$), take $m = n + a$, and

$$
\begin{aligned}
z_r &= 1 + m + n - \frac{1}{2}\left(|m - r - n| + |m + r - n|\right) \\
&= 1 + 2n + a - \frac{1}{2}\left(|a - r| + |a + r|\right) \\
&= 1 + 2n + a - \max(a, r)
\end{aligned}
.
$$

So, for $0 \leq r \leq a$, $z_r$ is constant at $1 + 2n$. In particular, $z_{-1} = z_{+1} = z_0$, so

$$\int_0^{\pi} z(\theta) w(\theta) d\theta = -\frac{1}{2} z_{-1} + z_0 - \frac{1}{2} z_1 = -\frac{1}{2} z_0 + z_0 - \frac{1}{2} z_0 = 0$$