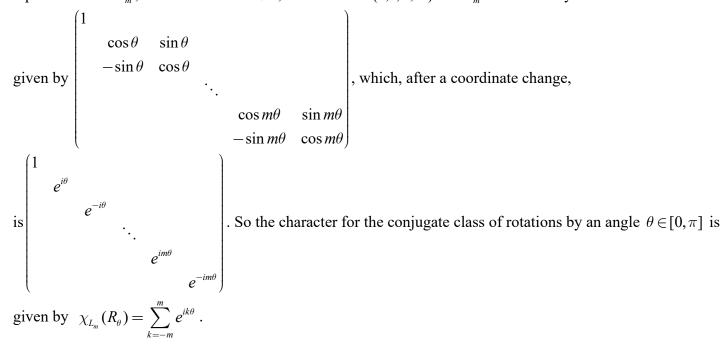
Linear Transformations and Group Representations

Homework #2 (2024-2025), Questions

Q1: Orthogonality of the characters of SO(3)

Setup: As detailed in class notes, the group of rotations of a sphere in 3-space, SO(3), has irreducible representations L_m , of dimension 2m+1, for each $m \in \{0,1,2,...\}$. In L_m . a rotation by θ about the "z" axis is



We further stated that these characters are orthonormal, if they are properly weighted by the "mass " of their conjugate classes: $\int_{0}^{\pi} \chi_{L_{m}}(R_{\theta}) \overline{\chi_{L_{n}}(R_{\theta})} w(\theta) d\theta = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$, where $w(\theta)$ is the weighting of the conjugate class θ , $w(\theta) = \frac{1}{\pi} (1 - \cos \theta)$. Here we demonstrate this orthonormality. Several steps but each should be straightforward.

- A. For any real-valued function $f(\theta)$ that can be written as $f(\theta) = \sum_{k=-m}^{m} f_k e^{ik\theta}$, determine $\int_{0}^{\pi} f(\theta) d\theta$ in terms of the f_k .
- B. Given any two functions that are of the form specified in part A (say, $f(\theta) = \sum_{k=-m}^{m} f_k e^{ik\theta}$ and $g(\theta) = \sum_{l=-n}^{n} g_l e^{il\theta}$), express the results of multiplying them as a third such function.
- C. Write $w(\theta) = \frac{1}{\pi} (1 \cos \theta)$ in the form specified in part A.

- D. For any non-constant real-valued function $f(\theta)$ of the form specified in part A, compute $\int_{0}^{n} f(\theta)w(\theta)d\theta$ in terms of the f_k .
- E. For $f(\theta) = \chi_{L_m}(R_{\theta})$, $g(\theta) = \chi_{L_n}(R_{\theta})$, and $z(\theta) = f(\theta)g(\theta)$, determine z_k so that $f(\theta)g(\theta) = \sum_{k=-(m+n)}^{m+n} z_k e^{ik\theta}$ and, with the results of part D, demonstrate orthonormality.