Biol. Cybern. 57, 421–426 (1987)

Biological Cybernetics

The Fractal Dimension of a Test Signal: Implications for System Identification Procedures

J. D. Victor

Department of Neurology, Cornell University Medical College, 1300 York Avenue, New York City, NY 10021, USA, and
Laboratory of Biophysics, The Rockefeller University, 1230 York Avenue, New York City, NY 10021, USA

Abstract. The experimental identification of a nonlinear biologic transducer is often approached via consideration of its response to a stochastic test ensemble, such as Gaussian white noise (Marmarelis and Marmarelis 1978). In this approach, the input-output relationship a deterministic transducer is described by an orthogonal series of functionals. Laboratory implementation of such procedures requires the use of a particular test signal drawn from the idealized stochastic ensemble; the statistics of the particular test signal necessarily deviate from the statistics of the ensemble. The notion of a fractal dimension (specifically the capacity dimension) is a means to characterize a complex time series. It characterizes one aspect of the difference between a specific example of a test signal and the test ensemble from which it is drawn: the fractal dimension of ideal Gaussian white noise is infinite, while the fractal dimension of a particular test signal is finite. This paper shows that the fractal dimension of a test signal is a key descriptor of its departure from ideality: the fractal dimension of the test signal bounds the number of terms that can reliably be identified in the orthogonal functional series of an unknown transducer.

Introduction

Much theoretical effort in neurobiology has been devoted to the development of an approach to the analysis of nonlinear transducers which is both satisfactory from a theoretical point of view, and amenable to laboratory implementation. Wiener's (1958) observation that a deterministic transducer can (in principle) be completely characterized by its response to a Gaussian white noise test ensemble led to one approach to this problem (for a review, see Marmarelis and Marmarelis 1978).

In reducing the Wiener approach to laboratory practice, one must generate a concrete approximation to a Gaussian white noise test ensemble. Unavoidable limitations of physical apparatus dictate that this test signal will not be of infinite bandwidth, and will not have a truly Gaussian amplitude distribution. Nevertheless, given the bandwidth limitations and saturations inherent in any biological transduction under study, it is possible to minimize the effects of this kind of nonideality (Marmarelis and Marmarelis 1978).

In the Wiener procedure, a transduction is represented by an orthogonal series of functionals. The n-th Wiener functional is specified by a symmetric function of n time-lags. In practice, one determines the value of this symmetric function at a mesh of points separated by a small time increment, rather at a continuum of times. Nevertheless, the number of degrees of freedom inherent in the n-th order Wiener kernel grows exponentially with n. Thus, a basic problem in the reduction of the Wiener procedure to laboratory practice stems from the need to determine a function of n variables (the n-th Wiener kernel) from a function of a single variable (the system's temporal response to the test signal). Intuitively, the ability to determine these kernels must rely on the "richness" of the test signal – i.e., its ability to explore a wide variety of possible inputs.

The concept of fractal dimension has recently been exploited in quantifying the complexity of physical systems (Mandelbrot 1977, 1982; Grassberger and Procaccia 1983; Mayer-Kress 1986). This concept may be applied to test signals as readily as it has been applied to the output, or spontaneous behavior, of physical systems. The point of this communication is that the capacity dimension [the prototypical fractal dimension (Farmer et al. 1983)] of a test signal specifies the "richness" of the signal in the sense discussed above: the n-th order Wiener kernel of an unknown transducer may be determined from its response to a particular "example" of Gaussian white noise at a given temporal discretization only if, at that level of discretization, the fractal dimension of the test signal is no less than n.

In the typical laboratory implementation of the Wiener procedure (Marmarelis and Marmarelis 1978), crude knowledge about a system's bandwidth and nonlinearity help specify an appropriate approximation to Gaussian white noise. Here we take the inverse approach: we ask how a particular approximation to Gaussian white noise limits the ability to identify nonlinear transducers.

Results

The Orthogonal Functional Approach

We begin by summarizing some aspects of the Wiener theory of nonlinear systems analysis. The reader is referred to Wiener (1958) or Marmarelis and Marmarelis (1978) for greater background and detail.

The objects of interest are causal, deterministic transducers μ. The response of a transducer μ to a test signal $s(t)$ at a time t will be denoted $\mu[s](t)$. Although the transducers are not necessarily linear, they still may be regarded as members of a linear vector space, with addition of two transducers defined by parallel combination

$$(\mu+v)[s](t)=\mu[s](t)+v[s](t) \tag{1}$$

and scalar multiplication defined by

$$(a\mu)[s](t)=a(\mu[s](t)). \tag{2}$$

We only consider transducers that are stationary in time and of finite memory. Time-stationarity is specified by

$$\mu[s_\tau](t)=\mu[s](t+\tau), \tag{3}$$

where

$$s_\tau(t)=s(t+\tau).$$

A transducer has finite memory of there exists a time interval $T_{\mathrm{mem}}(\mu)$ (the memory of the transducer μ) such that the response at any given time t depends only on values of the input at times more recent than $t-T_{\mathrm{mem}}(\mu)$:

$$\begin{aligned} &\mu[s](t)=\mu[s'](t) \quad \text{if} \\ &s(\tau)=s'(\tau) \quad \text{for} \quad \tau \leqq t-T_{\mathrm{mem}}(\mu). \end{aligned} \tag{4}$$

The Wiener formalism represents the input-output relationship of a transducer μ as an infinite sum of orthogonal functionals:

$$\mu=\sum_{n=0}^{\infty} \kappa_n. \tag{5}$$

Distinct functionals κ_m and κ_n of the Wiener series (5) are orthogonal in the sense that

$$\langle \kappa_m[s](0)\cdot \kappa_n[s](0)\rangle_{s\in\Omega}=0, \tag{6}$$

where $\langle\ \rangle_{s\in\Omega}$ indicates an average over all signals s of the ensemble Ω of Gaussian white noise test signals.

The n-th term in the orthogonal expansion (5) may be expressed as a functional Taylor series (Volterra 1932) of order n:

$$\kappa_n=\sum_{j=0}^{n} \kappa_{n,j}. \tag{7}$$

Each term $\kappa_{n,j}$ of (7) is a homogeneous transducer of order j. Its transduction is defined by a symmetric function $\{\kappa_{n,j}\}$ of time-lags:

$$\begin{aligned} \kappa_{n,j}[s](t)=&\int\int\ldots\int\{\kappa_{n,j}\}(\tau_1,\tau_2,\ldots,\tau_j) \\ &\times s(t-\tau_1)s(t-\tau_2)\cdot\ldots\cdot s(t-\tau_j)\,d\tau_1\,d\tau_2\ldots d\tau_j. \end{aligned} \tag{8}$$

Note that the Wiener orthogonal transducers κ_n contain a term $\kappa_{n,n}$ which is homogeneous of order n, as well as terms $\kappa_{n,j}$ $(j<n)$ of lower order. The lower-order terms are necessary to ensure that the components of the Wiener series (5) are orthogonal. Because of the even-symmetric properties of Gaussian white noise, the terms $\kappa_{n,j}$ of (7) are identically zero if n and j are of different parity.

The Wiener series (5) may be truncated after a finite number (say n) of terms. The truncated series is the best approximating transduction within the space spanned by homogeneous transducers up to order n. The approximating transduction is optimal in a mean-squared sense, when averaged over a test ensemble consisting of Gaussian white noise whose power per unit bandwidth is equal to that of the test signal. Because the approximation is optimal in a mean-squared sense, a truncated Wiener series may be regarded as the orthogonal projection of the transducer μ into the subspace of all transducers whose Volterra series terminates at order n.

The Wiener series (5) may be characterized by the Wiener functionals κ_n introduced at each order n of approximation. This functional is in turn described by the kernel of its highest-order homogeneous component $\{\kappa_{n,n}\}$; lower-order terms $\{\kappa_{n,j}\}$ of (7) are integrals over the kernels $\{\kappa_{n,n}\}$, in a manner which is determined by orthogonality constraints.

The most straightforward method for calculation of the kernel $\{\kappa_{n,n}\}$ from experimental data is the cross-correlation approach of Lee and Schetzen (1965). For "off-diagonal" terms (i.e., all τ_i distinct), the kernel $\{\kappa_{n,n}\}$ is given by

$$\begin{aligned} &\{\kappa_{n,n}\}(\tau_1,\tau_2,\ldots,\tau_n) \\ &\quad=\frac{1}{c_n}\langle s(-\tau_1)s(-\tau_2)\cdot\ldots\cdot s(-\tau_n)\mu[s](0)\rangle_{s\in\Omega}, \end{aligned} \tag{9}$$

where c_n is a normalization factor which depends on the power per unit bandwidth of the test signal. When two or more of the arguments τ_i of the n-th kernel $\{\kappa_{n,n}\}$ coincide, its value is given by an expression whose leading term is (9), but which contains additional terms as a consequence of the orthogonality constraints.

The key point for the present discussion is that, for any particular test signal $s(t)$, the values of the kernels κ_n in the Wiener series (5) are related in a *linear* fashion to the response $\mu[s](t)$. This is a corollary of the statement that the transducers κ_n define the projections of the transducer μ into a sequence of orthogonal subspaces: orthogonal projection is a linear transformation.

A Geometric Interpretation of the System-Identification Procedure

Exact evaluation of the Wiener kernels of an unknown transducer from input-output data requires an ensemble-average (9). In practice, one replaces this ensemble-average by a time-average over a particular sample input signal $s_0 \in \Omega$:

$$\{\kappa_{n,n}\}(\tau_1, \tau_2, \ldots, \tau_n) = \frac{1}{c_n}\langle s_0(t-\tau_1)s_0(t-\tau_2)\cdot \ldots \cdot s_0(t-\tau_n)\mu[s_0](t)\rangle_t . \tag{10}$$

The transformation from an ensemble-average (9) to a time-average requires not only the stationarity assumption (3), but also the assumption that the particular signal s_0 typifies the ensemble Ω. The second assumption is a kind of ergodicity, and is in principle well-justified. Difficulties occur in practice because the average in (10) is over a finite length of time. Furthermore, because of bandwidth limitations of both the stimulus and the transducer under study, only a finite number of values of $s_0(t)$ and $\mu[s_0](t)$ are independent.

We may view this situation in a geometric fashion. The test signal $s_0(t)$ traces a trajectory in the state space of possible inputs. The transducer μ defines a function on this trajectory, $\mu[s_0](t)$. Equation (10) shows how the n-th-order Wiener kernel may be extracted from the values of this function.

Conversely, for any given n-th-order Wiener kernel, one may construct a canonical transducer whose n-th order kernel is that specified symmetric function, by setting all other terms of (5) to zero. In order that the laboratory calculation of n-th-order kernels reflect the spirit of the Wiener formalism, it is necessary that the procedure (10) map the responses of each of these distinct canonical n-th order Wiener systems to linearly-independent kernel functions. Were a linear dependence of these kernels to exist, then it would be possible to construct distinguishable n-th-order systems with identical n-th-order kernels.

The calculation of the n-th-order kernel is fundamentally a map from functions defined on the trajectory of the test signal to symmetric functions $\{\kappa_{n,n}\}$, defined on n positive times τ_i. We will now show that the ability to extract a function of n variables (the kernel) from a function of a single variable (the transducer's response) places a lower bound of n on the capacity dimension of the test signal.

Procedures for kernel extraction that recognize the bandwidth- and amplitude-limited nature of any actual test signal $s_0(t)$ and that are more efficient computationally than the original Lee-Schetzen procedure are now available (Korenberg 1987). However, increases in computational efficiency do not change the fundamental geometric character of the mapping from transducer responses to Wiener kernels.

Connection with Fractal Dimension

To make the correspondence between limitations of practical implementations of the Wiener theory and the dimension of the test signal, two preliminaries are necessary (Mayer-Kress 1987). *(1)* One must recognize the difference between a formal definition of "dimension" as applied to a mathematical object, and as applied to laboratory data (i.e., the test signal). *(2)* The time-series of interest must be embedded in a high-dimension Euclidean space. We next consider these issues in more detail.

Dimension, as an attribute of an ideal mathematical object, describes a scaling property which holds as a parameter of fine-graining approaches zero. But, as applied to laboratory data, dimension describes scaling properties which hold only over a range in which the fine-graining parameter is bounded away from zero. In particular, the parameter of fine-graining must not be small in comparison to the scale of measurements, if the notion of dimension is to be meaningful.

Our parameter of fine-graining will be the time-interval at which the input and output signals are sampled. We will want to consider scaling properties as the sampling interval becomes small, but no smaller than some finite value $\Delta T_{\min}$. This finite value is determined by the bandwidth $1/\Delta T_{\min}$ of the laboratory approximation to Gaussian white noise, $s_0(t)$. Let us now fix a particular sampling interval ΔT. We consider those transducers whose kernels $\{\kappa_{n,n}\}$ are well-represented by discrete sampling at times $\tau_i = m_i \Delta T$. Such transducers will have a bandwidth no larger than (approximately) $1/\Delta T$. Since the test signal $s_0(t)$ has a higher bandwidth ($1/\Delta T_{\min}$), it appears "white" to the transducers under study. However, fluctuations of $s_0(t)$ on a timescale briefer than ΔT are irrelevant to these transducers. Thus, for analysis at a given temporal scale ΔT, we replace $s_0(t)$ by a filtered version whose bandwidth is of order $1/\Delta T$, and discretize the filtered signal in intervals ΔT.

For the chosen time-discretization ΔT, there is a set $K_n(\Delta T)$ of canonical transducers of order n. Each of these transducers has an n-th order kernel which is zero except at one particular set of (off-diagonal) time-lags $t_i = m_i \Delta T$, where it has the value of unity. Because of the symmetry of the Wiener kernel, we may assume that $t_1 < t_2 < \ldots < t_n$. For convenience, we write $\vec{t} = (t_1, \ldots, t_n)$. Thus, a typical element $\mu_{\vec{t}}$ of $K_n(\Delta T)$ has an n-th-order Wiener kernel $\{\kappa_{\vec{t}}\}_{n,n}$ given by

$$\{\kappa_{\vec{t}}\}_{n,n}(\tau_1, \tau_2, \ldots, \tau_n\} = \begin{cases} 1, & \tau_i = t_{\sigma(i)} \text{ for some permutation } \sigma; \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

We place a uniform limit T_{mem} on the time-lags t_i.

In order for a laboratory procedure based on the signal s_0 to provide for identification of n-th-order kernels of systems of bandwidth $1/\Delta T$, it is necessary for the kernels (11) of the canonical transducers to be linearly independent. If a linear relationship between the n-th-order kernels of transducers in $K_n(\Delta T)$ did exist, then there would be a (nonzero) n-th-order transducer whose n-th-order kernel was zero. Therefore, we use this criterion of linear independence as a test for the validity of a kernel extraction procedure based on s_0. We will see that this criterion ultimately places a limit on the dimension of the test signal s_0.

The second necessary step before invoking the dimension formalism is the embedding of the time-series of interest [in this case $s_0(t)$] into an M-dimensional Euclidean space $\mathbf{E}^M$. A well-established method for this embedding is to map the test signal $s_0(t)$ into an M-dimensional vector $\phi(t)$ given by time-delay coordinates. The time-delay ΔT_{lag} is typically chosen to be the first minimum of the mutual information content between $s_0(t)$ and $s_0(t - \Delta T_{\text{lag}})$ (Fraser and Swinney 1986). In the present context, the appropriate time-lag ΔT_{lag} is precisely the sampling interval:

$$\Delta T_{\text{lag}} = \Delta T. \tag{12}$$

Thus, we consider the embedding procedure

$$\phi(t) = (s_0(t), s_0(t - \Delta T), \ldots, s_0(t - (M-1)\Delta T)). \tag{13}$$

The embedding procedure (13) provides a smooth map from $s_0(t)$ into the embedding space $\mathbf{E}^M$. The time-scale ΔT now becomes a distance-scale in the embedding space $\mathbf{E}^M$: since the test signal $s_0(t)$ is both amplitude- and band-limited, the velocity $\dot{\phi}(t)$ is bounded. We choose units in $\mathbf{E}^M$ so that the typical velocity $|\dot{\phi}(t)|$ is unity.

Via the embedding map, each canonical transducer $\mu_{\vec{t}}$ defines a function on $\phi(t)$, namely $\mu_{\vec{t}}[s_0](t)$. Along the curve $\phi(t)$, this function is a smooth function of time. This is a consequence of the smoothness of $s_0(t)$ (for time intervals smaller than ΔT) and the defining properties (11) of $\mu_{\vec{t}}$. Furthermore, this smoothness is uniform across the set of canonical transducers $K_n(\Delta T)$, since $s_0(t)$ is amplitude-limited.

In general, it is necessary to ensure that the dimension M of the embedding space is sufficiently high: with an embedding dimension which is too low, the procedure (13) destroys information which is present in the original time-series, and the resulting estimate of dimension is too low (Mayer-Kress 1987). In the present context, preservation of relevant information by the embedding procedure requires that the embedding procedure respects the dynamics of the transducers under study. That is, if two points $\phi(t_1)$ and $\phi(t_2)$ are close in the embedding space $\mathbf{E}^M$, then a transducer's responses $\mu_{\vec{t}}[s_0](t_1)$ and $\mu_{\vec{t}}[s_0](t_2)$ must also be close. Preservation of transducer dynamics by the embedding procedure (13) is guaranteed if the embedding dimension M is sufficiently high so that the embedding includes all time-lags up to the maximum time-lag T_{mem} of the canonical transductions $K_n(\Delta T)$:

$$(M-1)\Delta T \geqq T_{\text{mem}}. \tag{14}$$

In summary, we have embedded a band-limited, amplitude-limited test signal $s_0(t)$ smoothly into a curve $\phi(t)$ in $\mathbf{E}^M$. The time-scale ΔT defines a sampling interval for $s_0(t)$ and for the Wiener kernels $\{\kappa_{\vec{t}}\}_{n,n}$, as well as a distance scale on $\mathbf{E}^M$. For any canonical transducer μ in $K_n(\Delta T)$, the smooth dependence of $\mu[s_0](t)$ on time translates into a smooth dependence on points in $\mathbf{E}^M$. In the context of the discretization, $\mu[s_0](t_1)$ and $\mu[s_0](t_2)$ may be chosen independently only if

$$|\phi(t_1) - \phi(t_2)| > \Delta T. \tag{15}$$

With this machinery in place, we may calculate the capacity dimension of the signal $s_0(t)$. The capacity dimension (Farmer et al. 1983) describes how the number of spheres $N(\varepsilon)$ of a given size ε required to cover a set scales with the size of the sphere. For a set of capacity dimension d_{cap},

$$N(\varepsilon) \propto \varepsilon^{-d_{\text{cap}}} \quad \text{as} \quad \varepsilon \to 0. \tag{16}$$

Consider a covering of the embedded trajectory $\phi(t)$ with a minimum number $N(\Delta T)$ spheres of size ΔT. We now use the hypothesis that the transducers $\mu_{\vec{t}}$ have linearly independent kernels to deduce a lower bound for $N(\Delta T)$. Each transducer $\mu_{\vec{t}}$ assigns a value $\mu_{\vec{t}}[s_0](t)$ to the sphere centered at t. The kernels are linear functions of these assigned values, since (at the scale ΔT of discretization), the value assigned by a given transducer is constant within spheres of radius ΔT. Therefore, in order for the kernels to be linearly independent, there must be at least as many spheres as there are transducers in the canonical set $K_n(\Delta T)$.

The number of transducers in $K_n(\Delta T)$ is equal to the number of ordered n-tuples of time-lags $\vec{t}=(t_1, \ldots, t_n)$, where each t_i is a multiple of ΔT and is no larger than T_{mem}. Thus, there are $\frac{1}{n!}(T_{\mathrm{mem}}/\Delta T)^n$ (plus terms of lower order in $(T_{\mathrm{mem}}/\Delta T)$) distinct transducers in $K_n(\Delta T)$. The number of spheres $N(\Delta T)$ in a covering of $\phi(t)$ must be at least as large as the number of transducers in $K_n(\Delta T)$, since we have hypothesized that these canonical transducers have linearly-independent kernels. That is,

$$N(\Delta T) \geqq \frac{1}{n!}\left(\frac{T_{\mathrm{mem}}}{\Delta T}\right)^n. \tag{17}$$

Thus, if the canonical n-th-order Wiener transductions $K_n(\Delta T)$ are linearly independent, the apparent capacity dimension of a test signal $s_0(t)$ is no less than n.

Discussion

The main point of this communication is the connection between the dimension formalism and the Wiener orthogonal functional method of nonlinear systems analysis. The dimension formalism, as applied to laboratory data, requires numerical approximations to what is in principle a limiting process; these approximations correspond in a natural way to the approximations made in a laboratory implementation of the Wiener theory. With these correspondences, we showed that the capacity dimension of a test signal limits the number of terms of the Wiener series that may be extracted.

We have not presented a formal proof of this relationship; such a proof would require a variety of technical assumptions concerning the manner in which a sequence of laboratory transductions approached Gaussian white noise, as well as a formal definition of dimension as applied to a real (laboratory) signal. Instead, we have focussed on the correspondences between Wiener analysis and dimension calculations, when both techniques are applied to real data.

It is well-recognized that the bandwidth limitations inherent in any realization of Gaussian white noise must be taken into account when applying the Wiener technique to a real transducer (Marmarelis and Marmarelis 1978). Fractal dimension, however, provides a measure of the complexity of a time series, which is independent of its spectral characteristics (Mayer-Kress and Layne 1986). Ideal Gaussian white noise has infinite fractal dimension. However, it is possible to generate signals with white spectra but which have low fractal dimension. Indeed, the sequence produced by a congruential random-number generator has a fractal dimension of 1. Conversely, bandpass-filtered transformations of ideal Gaussian white noise still have infinite fractal dimension.

Thus, fractal dimension in some sense characterizes the richness of a time series: i.e., the dimension of the phase space that it samples. This is why the dimension of the test signal, and not its bandwidth, limits the number of terms in a Wiener series that may be properly calculated. In view of this fact, a test signal's dimension, as well as its spectrum and amplitude distribution, should be used as an a priori criterion for the validity of a Wiener procedure.

Several extensions of the Wiener approach have been proposed (Yasui 1979; Victor and Knight 1979; Sutter 1987). These extensions share the notion of orthogonal expansion with respect to a stimulus ensemble, but differ as to choice of the ensemble. The association between kernel calculation and dimension of the test signal relies fundamentally on the linear relationship between the transducer's response and the kernel values. This relates the number of linearly-independent canonical kernels, and the number of independent points in phase space sampled by the test signal. This main feature of the original Wiener approach is shared by any orthogonal-functional technique. Thus, the significance of the fractal dimension of the test signal extends to other orthogonal-functional techniques as well.

Some of these extensions are based on complex but deterministic signals. In the M-sequence technique of Sutter (1987), the test signal is a sequence of binary levels drawn from a maximum-length shift register sequence of order n. The natural embedding for an M-sequence is a space of time delay coordinates, with the number of lags equal to the order of the shift register sequence. The properties of the M-sequence guarentee that each of the 2^n lattice points is sampled exactly once, except for the origin, which is unsampled. In the sum-of-sinusoids technique (Victor and Knight 1979), the test signal is a sum of n sinusoids whose frequencies are chosen to be large integer multiples of a common fundamental. The natural embedding space for this test signal is an n-torus, with each coordinate assigned to the phase of one component sinusoid. The trajectory of such a signal is a closed curve that samples the n-torus uniformly. (In the limit that the frequencies of the sinusoids are incommensurate, the trajectory would be dense in this space.)

While stochastic techniques *in principle* provide for measurement of kernels of all orders, in practice measurement is limited by the effective dimension of the test signal to a finite order. In general, the sampling, of phase space provided by a stochastic signal is irregular and nonuniform. In the deterministic techniques, phase space of a predetermined dimension is sampled in a homogeneous fashion. This distinction should be borne in mind when choosing an experimental approach to a system-identification problem.

Acknowledgements. The author thanks G. Mayer-Kress for discussions, and is indebted to V. Marmarelis and the Biomedical Simulations Resource for organizing the 1987 Workshop at which these discussions occurred. J. Victor is supported in part by NIH grants EY6871 and NS877, The Hartford Foundation, and the McKnight Foundation.

References

Farmer JD, Ott E, Yorke JA (1983) The dimension of chaotic attractors. Physica 7D:153–180

Fraser AM, Swinney HL (1986) Using mutual information to find independent coordinates for strange attractors. Phys Ref A33:1134–1140

Grassberger P, Procaccia I (1983) Measuring the strangeness of strange attractors. Physica 9D:189–208

Korenberg M (1987) Exact orthogonal kernel estimation from finite data records: extending Wiener's identification of nonlinear systems. Ann Biomed Eng (in press)

Lee YN, Schetzen M (1965) Measurement of the kernels of a nonlinear system by cross-correlation. Int J Control 2:237–254

Mandelbrot B (1977) Fractals: form, chance, and dimension. Freeman, San Francisco

Mandelbrot B (1982) The fractal geometry of nature. Freeman, San Francisco

Marmarelis PZ, Marmarelis VZ (1978) Analysis of physiological systems: the white-noise approach. Plenum Press, New York

Mayer-Kress G (1986) (ed) Dimensions and entropies in chaotic systems. Springer, Berlin Heidelberg New York

Mayer-Kress G (1987) Application of dimension algorithms to experimental chaos. Los Alamos preprint LA-UR-87-1030, Los Alamos National Laboratory, Los Alamos

Mayer-Kress G, Layne S (1986) Dimensionality of the human electroencephalogram. Los Alamos preprint LA-UR-86-1664, Los Alamos National Laboratory, Los Alamos

Sutter E (1987) A practical nonstochastic approach to nonlinear time-domain analysis. In: Marmarelis V (ed) Advanced methods of physiological system modelling. Biomedical Simulations Resource, Los Angeles

Victor JD, Knight BW (1979) Nonlinear analysis with an arbitrary stimulus ensemble. Q Appl Math 37:113–136

Volterra V (1932) Theory of functionals and of integral and integro-differential equations. Blackie, London Glasgow

Wiener N (1958) Nonlinear problems in random theory. Wiley, New York

Yasui S (1979) Stochastic functional Fourier series, Volterra series, and nonlinear systems analysis. IEEE Trans AC-24:230–242

Received August 6, 1987

Dr. Jonathan D. Victor
Department of Neurology
Cornell University Medical College
1300 York Avenue
New York City, NY 10021
USA

Verantwortlich für den Textteil: Prof. Dr. W. Reichardt, Max-Planck-Institut für biologische Kybernetik, Spemannstr. 38, D-7400 Tübingen. Verantwortlich für den Anzeigenteil: E. Lückermann, Springer-Verlag, Heidelberger Platz 3, D-1000 Berlin 33, Fernsprecher: (030)8207-0, Telex: 185411.
Druck der Brühlschen Universitätsdruckerei, Gießen. Printed in Germany. – Springer-Verlag GmbH & Co KG, 1000 Berlin 33