Calculation of the Minkowski image functionals on lattices with different connectivities

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This details the calculations for equations (1) through (6) in Victor, J.D., Thengone, D.J., Rizvi, S.M., and Conte, M.M. (2015) A perceptual space of local image statistics. Vision Research 117, 117-135.

Each 2x2 block is considered locally, with each "1" taken as a point (vertex), and lines connected 1's considered as "edges", and enclosed regions as "faces". This convention, with 1 representing black material and 0 representing white empty space, corresponds to that of Michielsen, K., & De Raedt, H. (2001), Physics Reports-Review Section of Physics Letters, 347(6), 462–538. But the assignment of 1 to black and 0 to white is the opposite of the convention used elsewhere in the VR paper. If one reverses the color-number convention, then the signs of γ and the θ's are inverted, which is equivalent to replacing $\chi^{[4]}$ by $-\chi^{[8]}$, $\chi^{[8]}$ by $-\chi^{[4]}$, and inverting the signs of the $\chi^{[6]}$. This does not change the spaces spanned or the results of the paper.

All vertices are 4-shared, since each pixel is part of 4 2x2 arrays, so the number of vertices, V, contributes $V/4$ to the Euler characteristic.

For edges, most are on the edges of the 2x2 array, so they are shared with another 2x2 array, and hence, if there are E_2 of them, they contribute $E_2/2$. A few edges are internal to a 2x2 array, and if there are a total of E_1 of them, they contribute $E_1/1=E_1$. Faces are all internal to a 2x2 array (since they are bounded by edges) and therefore are 1-shared. So $\chi = V/4 - E_2/2 - E_1 + F$, where V is the number of vertices, E_1 is the number of 1-shared edges, E_2 is the number of 2-shared edges, and F is the number of faces.

Config	number of equiv (k)	4-coordinated						8-coordinated						6-coordinated (6L)					
		$\begin{matrix} & N & \\ N & * & N \\ & N & \end{matrix}$						$\begin{matrix} N & N & N \\ N & * & N \\ N & N & N \end{matrix}$						$\begin{matrix} N & N & \\ N & * & N \\ & N & N \end{matrix}$					
		V	E_1	E_2	F	χ	$k\chi$	V	E_1	E_2	F	χ	$k\chi$	V	E_1	E_2	F	χ	$k\chi$
0 0 0 0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1 0 0 0	4	1	0	0	0	1/4	1	1	0	0	0	1/4	1	1	0	0	0	1/4	1
1 1 0 0	4	2	0	1	0	0	0	2	0	1	0	0	0	2	0	1	0	0	0
1 0 0 1	1	2	0	0	0	1/2	1/2	2	1	0	0	-1/2	-1/2	2	1	0	0	-1/2	-1/2
0 1 1 0	1	2	0	0	0	1/2	1/2	2	1	0	0	-1/2	-1/2	2	0	0	0	1/2	1/2

1 1 1 0	2	3	0	2	0	-1/4	-1/2	3	1	2	1	-1/4	-1/2	3	0	2	0	-1/4	-1/2
1 1 0 1	2	3	0	2	0	-1/4	-1/2	3	1	2	1	-1/4	-1/2	3	1	2	1	-1/4	-1/2
1 1 1 1	1	4	0	4	0	1	0	4+1*	4	4	4	0	0	4	1	4	2	0	0

Notes: * indicates that there is a fifth vertex, one-shared, that is formed in the middle by a crossing.
Also, there's an obvious mirror-flip of the 6L-connected lattice, i.e., 6R. Same calculation except that the two diagonal 2x2's play opposite roles (which makes a difference), and the two kinds of three-1 blocks play opposite roles (which makes no difference to the Euler characteristic).

Using Table 1 of Victor & Conte (2012) to convert block probabilities to coordinates:

$$\chi^{[4]}=\frac{1}{4}\left(p\begin{pmatrix}1&0\\0&0\end{pmatrix}+p\begin{pmatrix}0&1\\0&0\end{pmatrix}+p\begin{pmatrix}0&0\\1&0\end{pmatrix}+p\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)+\frac{1}{2}\left(p\begin{pmatrix}1&0\\0&1\end{pmatrix}+p\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$$
$$-\frac{1}{4}\left(p\begin{pmatrix}1&1\\1&0\end{pmatrix}+p\begin{pmatrix}0&1\\1&1\end{pmatrix}+p\begin{pmatrix}1&1\\0&1\end{pmatrix}+p\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)$$

$$\chi^{[8]}=\frac{1}{4}\left(p\begin{pmatrix}1&0\\0&0\end{pmatrix}+p\begin{pmatrix}0&1\\0&0\end{pmatrix}+p\begin{pmatrix}0&0\\1&0\end{pmatrix}+p\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)-\frac{1}{2}\left(p\begin{pmatrix}1&0\\0&1\end{pmatrix}+p\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$$
$$-\frac{1}{4}\left(p\begin{pmatrix}1&1\\1&0\end{pmatrix}+p\begin{pmatrix}0&1\\1&1\end{pmatrix}+p\begin{pmatrix}1&1\\0&1\end{pmatrix}+p\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)$$

$$\chi^{[6L]}=\frac{1}{4}\left(p\begin{pmatrix}1&0\\0&0\end{pmatrix}+p\begin{pmatrix}0&1\\0&0\end{pmatrix}+p\begin{pmatrix}0&0\\1&0\end{pmatrix}+p\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)-\frac{1}{2}p\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{1}{2}p\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$
$$-\frac{1}{4}\left(p\begin{pmatrix}1&1\\1&0\end{pmatrix}+p\begin{pmatrix}0&1\\1&1\end{pmatrix}+p\begin{pmatrix}1&1\\0&1\end{pmatrix}+p\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)$$

$$\chi^{[4]}+\chi^{[8]}=\chi^{[6L]}+\chi^{[6R]}=$$
$$\frac{1}{2}\left(p\begin{pmatrix}1&0\\0&0\end{pmatrix}+p\begin{pmatrix}0&1\\0&0\end{pmatrix}+p\begin{pmatrix}0&0\\1&0\end{pmatrix}+p\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)-\frac{1}{2}\left(p\begin{pmatrix}1&1\\1&0\end{pmatrix}+p\begin{pmatrix}0&1\\1&1\end{pmatrix}+p\begin{pmatrix}1&1\\0&1\end{pmatrix}+p\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)=.$$
$$\frac{1}{8}\left(-4\gamma+\theta_{\lrcorner}+\theta_{\llcorner}+\theta_{\ulcorner}+\theta_{\urcorner}\right)$$

$$\chi^{[4]}-\chi^{[8]}=p\begin{pmatrix}1&0\\0&1\end{pmatrix}+p\begin{pmatrix}0&1\\1&0\end{pmatrix}=\frac{1}{8}\left(1-2\beta_{-}-2\beta_{|}+\beta_{/}+\beta_{\backslash}+\alpha\right).$$

$$\chi^{[6L]}-\chi^{[6R]}=-p\begin{pmatrix}1&0\\0&1\end{pmatrix}+p\begin{pmatrix}0&1\\1&0\end{pmatrix}=\frac{1}{8}\left(-\theta_{\lrcorner}+\theta_{\llcorner}+\theta_{\urcorner}-\theta_{\ulcorner}\right).$$