Functions confined in space and spatial frequency include Gabor-like functions as well as intrinsically two-dimensional functions resembling "non-Cartesian" gratings

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SUMMARY

Established theoretical reasons recommend functions that are limited in space and spatial frequency as models for receptive fields of visual neurons. On the basis of an interpretation of spread as variance and via analogy to quantum mechanics, the Gabor functions are often stated to be maximally limited in space and spatial frequency (Daugman, 1985). Some Gabor functions indeed resemble receptive fields in V1, but this view provides little insight into why V1 receptive fields have only a small number of lobes, nor into receptive field shape beyond V1.

Here we consider an alternative interpretation of "limited in space and spatial frequency." We consider a function to be "confined" in space (or spatial frequency) if it is unchanged by windowing in space (or spatial frequency). While no function can be simultaneously confined in both space and spatial frequency, there is a rigorous sense in which the 2-dimensional Hermite functions achieve simultaneous confinement as nearly as possible. The 2-dimensional Hermite functions are a complete basis set and form a natural hierarchy. The first levels of this hierarchy contain functions that resemble Gabor functions with a small number of lobes, and thus resemble V1 receptive fields. Further down the hierarchy are intrinsically 2-dimensional functions, some of which resemble the non-Cartesian gratings, to which some V4 neurons respond preferentially (Gallant et al., 1996).

In addition to their many interesting mathematical properties, the two-dimensional Hermite functions allow for efficient ("sparse") local synthesis of images, including natural scenes, faces, and letters.

While we make no claim that this view suffices to account for receptive field structure, we suggest that it provides a framework for a principled study of receptive fields, and that it is useful to think of receptive fields (along the V1-to-V2-to-V4 pathway) as not only expanding, but also increasing in their combined space-bandwidth aperture.

BIOLOGICAL MOTIVATIONS

•Receptive fields do not merely enlarge from V1 to V2 to V4 (and beyond); they also acquire new properties. But it is likely that their structure is influenced by a common set of principles.

•Processing of objects relies on spatial frequency bands *scaled to the object*, e.g., 3 cycles/object for letters, 6 to 8 cycles/object for faces. This suggests consideration of constraints that are scale-invariant.

IDENTIFYING FUNCTIONS LIMITED IN SPACE AND SPATIAL FREQUENCY

Traditional approach: minimize variance

Daugman (1985)

Spatial spread Δx, Δy of a sensitivity profile $f(x,y)$:

Find the centroid (x_0,y_0) of $f(x,y)$.

Spread in x-direction, Δx: $(\Delta x)^2 = \iint (x-x_0)^2 |f(x,y)|^2 dxdy$

Spread in y-direction, Δy: $(\Delta y)^2 = \iint (x-y_0)^2 |f(x,y)|^2 dxdy$

Spatial frequency spread Δw_x, Δw_y of $f(x,y)$:

Determined analogously from the Fourier transform $\tilde{f}(w_x,w_y)$ of $f(x,y)$.

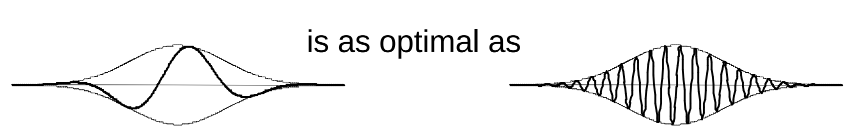
The Gabor functions minimize the product $\Delta x\, \Delta y\, \Delta w_x\, \Delta w_y$.

Concerns

This notion of minimization relies on complex nature of f: A Gabor function is a product of a complex exponential and a Gaussian:

$$f(x,y) = e^{i(w_x x + w_y y + j)} e^{\left(\frac{(x-x_0)^2}{2s_x^2} + \frac{(y-y_0)^2}{2s_y^2}\right)}.$$

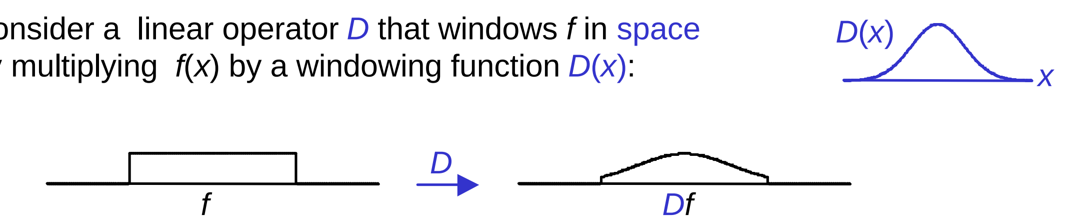
The real and imaginary parts together constitute an optimum, but neither is optimal alone. The minimization of the variance product depends only on $|f|$; the carrier frequency $w=(w_x, w_y)$ is irrelevant. In one spatial dimension,



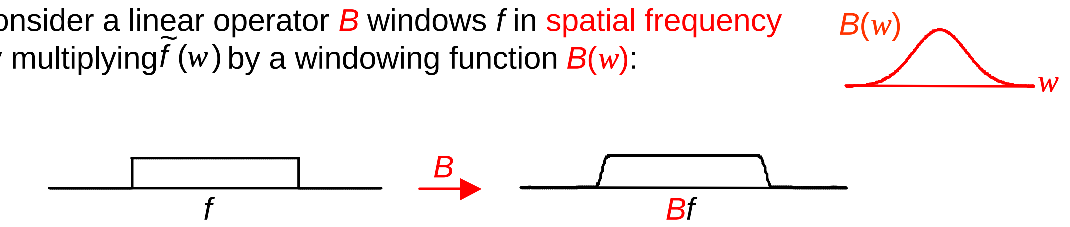
In two spatial dimensions, the orientation of the envelope and the carrier need not be related.

Novel approach: maximize confinement

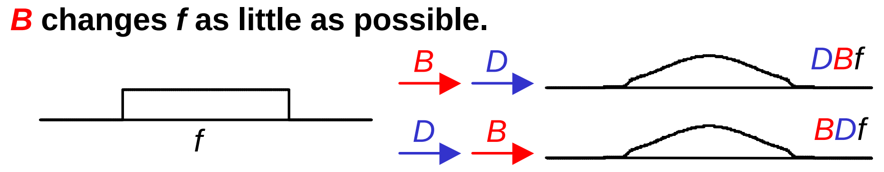
Consider a linear operator D that windows f in space by multiplying $f(x)$ by a windowing function $D(x)$:



Consider a linear operator B windows f in spatial frequency by multiplying $\tilde{f}(w)$ by a windowing function $B(w)$:

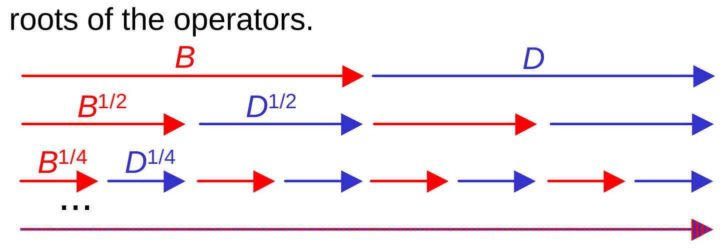


Functions f that are confined in space are unchanged after application of D. Functions f that are confined in *spatial frequency* are unchanged after application of B. **We therefore seek functions f for which successive application of D and B changes f as little as possible.**



Concerns

Since B and D do not commute, the order of application matters. But this potential difficulty can be avoided by considering successive application of infinitesimal "slices" of B and D. Infinitesimals can be found by taking successive square roots of the operators.

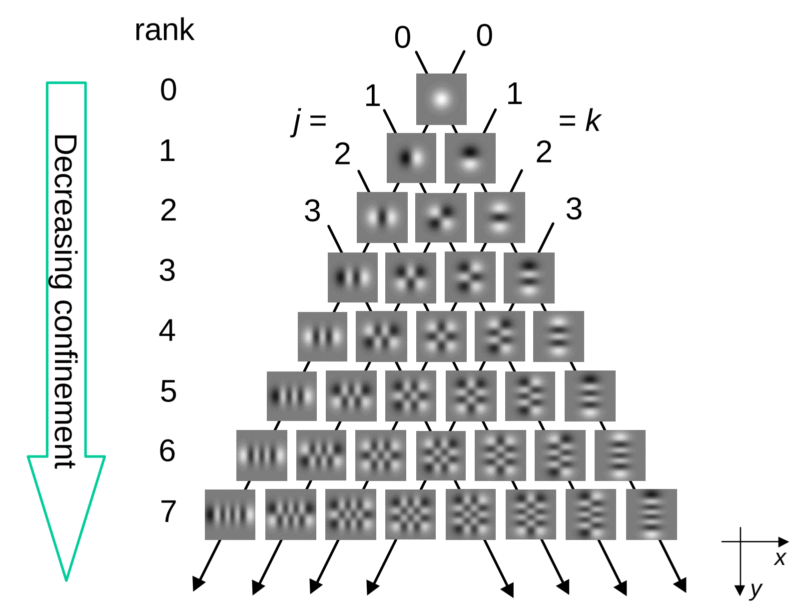


The second concern is that the optimal f's might depend on the shapes chosen for D and B. However, in limit of large* (space)(bandwidth) product, the optimal f's are independent of this choice. In this asymptotic limit, the optimal functions have a simple closed form: the two-dimensional Hermite functions.

*Large: two cycles per spatial aperture suffice (4π>>1)

TWO-DIMENSIONAL HERMITE FUNCTIONS

Cartesian separation



•Each Cartesian-separated function (*left*) is a product of a Hermite function $h_j(x)$, and a Hermite function $h_k(y)$. The rank is $j+k$.

•The rank 0 function ($j=k=0$ in the Cartesian separation) is a two-dimensional Gaussian. It is optimally confined in space and spatial frequency (in the above sense).

•The functions of each rank are the most confined functions *that are orthogonal to all functions of previous ranks.*

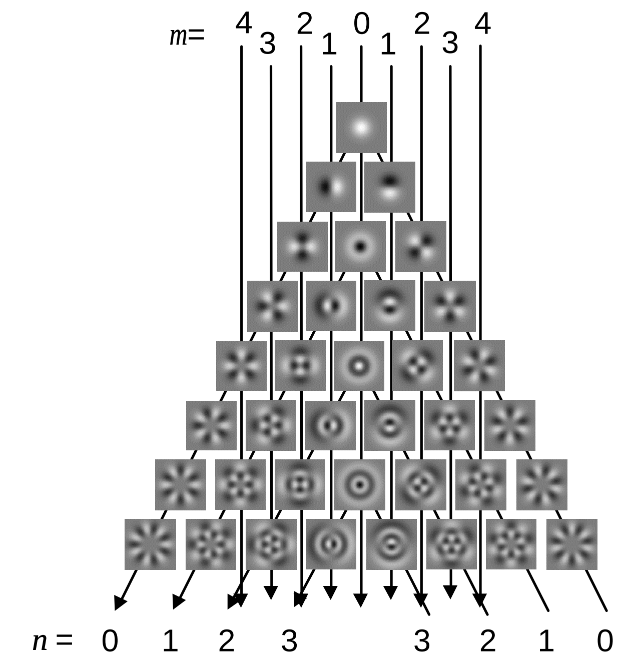
•The full set of functions constitutes a complete basis set.

•Arbitrary linear combinations of the Cartesian functions within each rank share the orthogonality property (between ranks) and the optimal-confinement property.

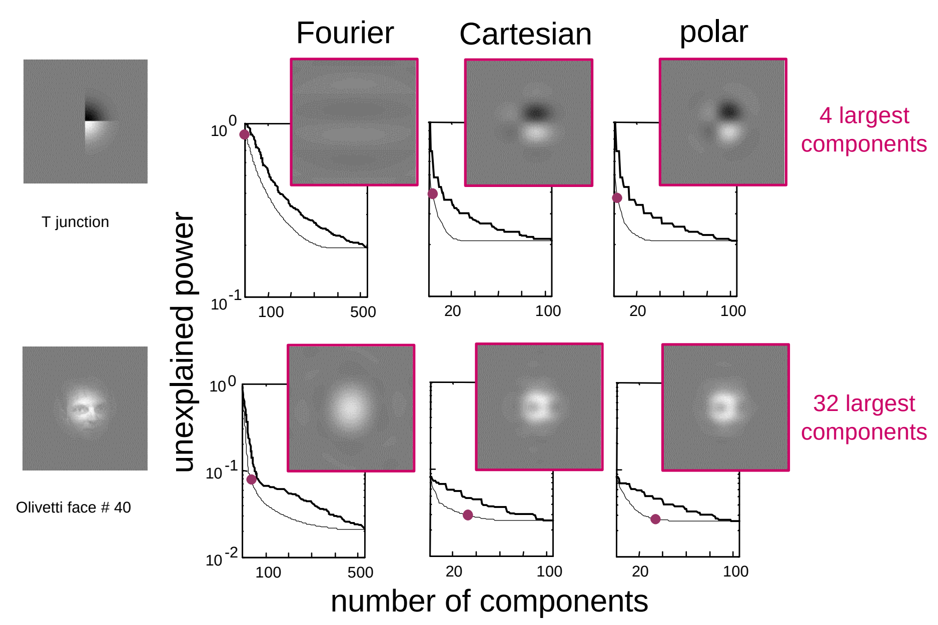
•A particular set of linear combinations can be formed that respects polar symmetry. These polar-separated functions (*right*) are parameterized by their rotational symmetry, m, and the number of radial nodes, n. Rank is $m+2n$. For m>0, the functions form sine and cosine pairs.

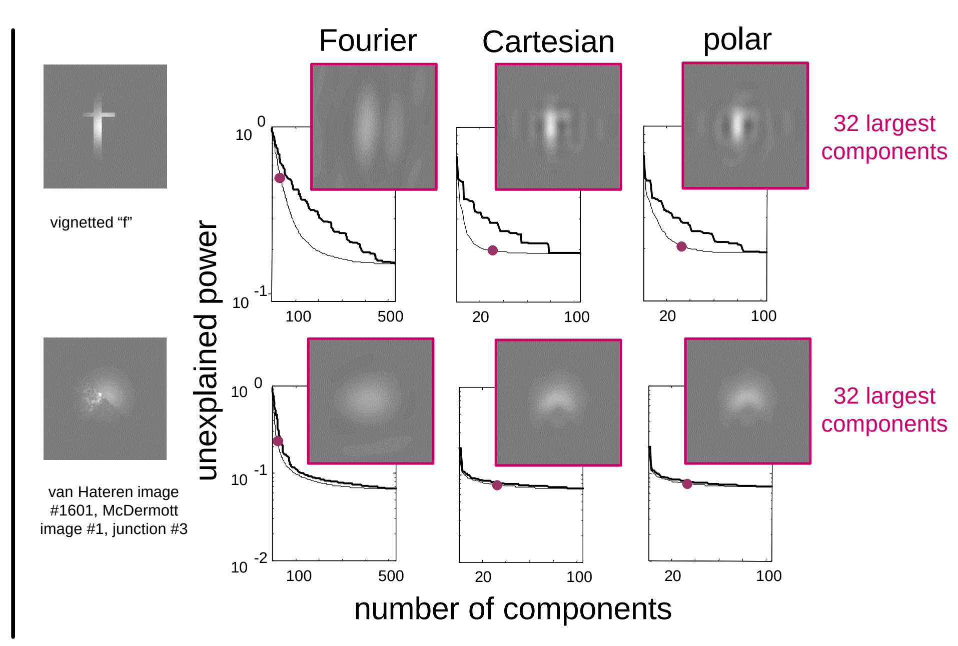
•Both the Cartesian-separated set of functions and the polar-separated set of functions are orthogonal within ranks, as well as across ranks.

Polar separation



EFFICIENT LOCAL SYNTHESIS





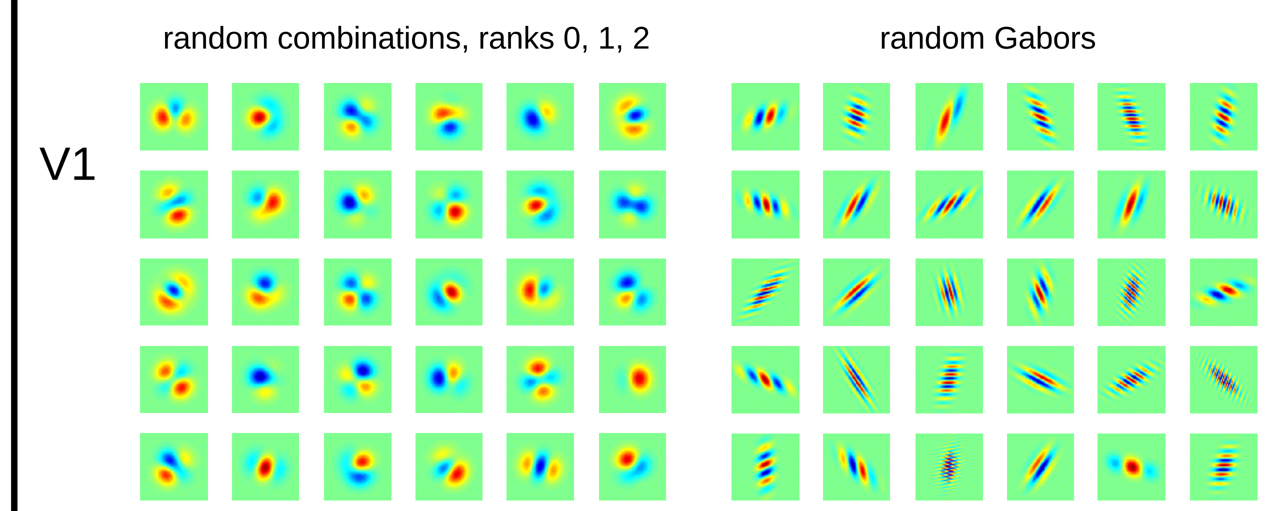
Quantification of sparseness. The index I_{conc} indicates the extent to which the N weights r_i are concentrated at a small number of basis functions:

$$I_{conc} = \sum_{i=1}^{N} (r_i)^4 \Big/ \left[\sum_{i=1}^{N} (r_i)^2\right]^2$$

(kurtosis $g = NI_{conc} - 3$)

	Fourier	Cartesian	polar
T-junction	0.009	0.239	0.242
face	0.050	0.887	0.889
letter	0.013	0.223	0.214
natural T-junction	0.049	0.745	0.745

A STARTING POINT FOR RF MODELS?



too blobby for V1; some bizarre shapes

carrier and envelope disproportionate and of differing orientations

bizarre shapes, few symmetric

more symmetric, but why limit rank?

CONCLUSIONS

•A novel (but natural) notion of simultaneous confinement in space and spatial frequency leads to a set of functions that are not Gabor functions.

•Low-rank two-dimensional Hermite functions are similar to Gabor functions with a small number of lobes. Higher-rank functions explore two-dimensional aspects of images, and do not resemble Gabor functions with many lobes.

•Two-dimensional Hermite functions form a complete basis set with convenient analytic properties. They are balanced in spatial frequency, but diverse in symmetry properties.

•First principles alone do not account for receptive field profiles in V1 through V4, but do suggest a rational basis for studying them.

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