1

Ensembles, Estimators, Gaussians HW 1 Answers $\boxed{1}$ – $\boxed{10}$

1. $$\hat{p}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} p(x)\,dx = \frac{1}{\pi a}\int_{-\infty}^{\infty} \frac{e^{-i\omega x}\,dx}{1 + x^2/a^2}$$

$$= \frac{1}{2\pi a}\int_{-\infty}^{\infty} e^{-i\omega x}\left(\frac{1}{1 - ix/a} + \frac{1}{1 + ix/a}\right) dx.$$

Recall: F.T. of $\frac{1}{\lambda} e^{-t/\lambda} = \frac{1}{1 - i\omega\lambda}$, i.e.,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}\,d\omega}{1 - i\omega\lambda} = \begin{cases} \frac{1}{\lambda} e^{-t/\lambda}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

$$\therefore \frac{1}{2\pi a}\int_{-\infty}^{\infty} \frac{e^{-i\omega x}\,dx}{1 + ix/a} = \begin{cases} e^{-\omega a}, & \omega \geq 0 \\ 0, & \omega < 0. \end{cases} \qquad \text{"A"}$$

$$\frac{1}{2\pi a}\int_{-\infty}^{\infty} \frac{e^{-i\omega x}\,dx}{1 - ix/a} = \begin{cases} 0 & \omega > 0 \\ e^{\omega a} & \omega \leq 0 \end{cases} \qquad \text{"B"}$$

So $\hat{p}(\omega) = \text{"A"} + \text{"B"} = e^{-|\omega| a}$. Note $\hat{p}'(0)$, $\hat{p}''(0)$ don't exist.

$\hat{p}_a(\omega) * \hat{p}_b(\omega) = \hat{p}_{a+b}(\omega)$; CLT does not hold.

Also, $\langle x^2 \rangle = \int_{-\infty}^{\infty} \frac{1}{\pi a} \frac{x^2\,dx}{1 + x^2/a^2}$ does not exist.

Ensembles, Estimators, Gaussian HW1 Ans 1 - 10

Notes: Pareto - Levy "stable" distribution.

Say $\hat{\rho}_a(\omega) = e^{-|\omega|^\gamma a}, \quad 0 < \gamma \le 2.$

Then $\hat{\rho}_a(\omega) * \hat{\rho}_b(\omega) = e^{-|\omega|^\gamma (a+b)} = \hat{\rho}_{a+b}(\omega),$

so these are stable (in shape) under convolution.
For $\gamma < 2$, $\langle x^2 \rangle$ is infinite.
[For $\gamma > 2$, $\rho(x)$ is not always positive]

2. $C_\rho(\omega) = \log(\hat{\rho}(\omega)) = -\frac{\omega^2}{2}\langle x^2\rangle + \frac{\omega^4}{24}\langle x^4\rangle$

$\left[\langle x\rangle = \langle x^3\rangle = 0\right]$ $\quad -\frac{1}{2}\left(-\frac{\omega^2}{2}\langle x^2\rangle + \frac{\omega^4}{24}\langle x^4\rangle\right)^2$

$+\frac{1}{3}\left(-\frac{\omega^2}{2}\langle x^2\rangle + \frac{\omega^4}{24}\langle x^4\rangle\right)^3 \ldots$

$$-\frac{A_2\omega^2}{2!} + \frac{A_4\omega^4}{4!}\ldots = -\frac{\langle x^2\rangle}{2}\omega^2 + \left(\frac{\langle x^4\rangle}{24} - \frac{\langle x^2\rangle^2}{2\cdot 2^2}\right)\omega^4\ldots$$

$$A_2 = \langle x^2\rangle$$

$$A_4 = \langle x^4\rangle - 3\langle x^2\rangle^2 \qquad \text{kurtosis} = \frac{A_4}{(A_2)^2}.$$