☐ Eigenvalues, Eigenvectors, + Inner Products

L in $\mathrm{Hom}(V,V)$, V of dimension $n < \infty$

char. eq'n of $L = \det(L - zI)$

If k is algebraically closed, $\det(L-zI) = (\lambda_1 - z)(\lambda_2 - z)\cdots(\lambda_n - z)$

Each term of the c.e. is a symmetric function of the eigenvalues

coef of z^0 = product of eigenvalues = $\det L$
coef of $z^1 = -(\lambda_2\lambda_3\lambda_4\cdots\lambda_n) - (\lambda_1\lambda_3\lambda_4\cdots\lambda_n) \cdots - (\lambda_1\lambda_2\cdots\lambda_{n-1})$
$\vdots$
coef of $z^{n-1} = (-1)^{n-1}(\lambda_1 + \cdots + \lambda_n)$
coef of $z = (-1)^n$

Define trace: $\mathrm{tr}(L)$ = coef of $(-1)^{n-1} z^{n-1}$
Manifestly coordinate-independent.
But in coords, $\mathrm{tr}(L)$ = sum of diagonal elements

Since $\mathrm{tr}(ALA^{-1}) = \mathrm{tr}(L)$ [adjoint action, coord-indep]
(1) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ [$B = LA^{-1}$]

(2) tr is a linear map from $\mathrm{Hom}(V,V)$ to k.
Above properties (1), (2) define tr, up to a scalar factor.

Note $\mathrm{tr}(AB) \neq \mathrm{tr}(A)\,\mathrm{tr}(B)$.

Commuting operators + eigenvectors [operator ≡ linear transf.]

Say $A, B \in \mathrm{Hom}(V,V)$, and $AB = BA$, and
$Av = \lambda v$.

Then $A(Bv) = B(Av) = B\lambda v = \lambda(Bv)$
So, Bv is also an eigenvector for A, with eiv λ.

For the "typical" operator, in a finite-dim. space,
each eigenspace is 1-dimensional + is associated with a distinct eigenvalue. [characteristic eq'n has distinct roots;
if $\det(A - \lambda I) = 0$, then
$Av - \lambda v = 0$ for <u>some</u> v;
distinct eigen~~values~~ vectors must have distinct eigenvalues
since $\lambda v = \mu v$ only if $v = 0$;
counting argument shows that there are n distinct eigenspaces of dimension 1]

For the "typical" operator,
the eigenvectors form a basis (unique up to individual scalar multiplication),
so if Bv and v are both eigenvectors $\bar{c}$ eiv λ,
it follows that $Bv = \mu v$, and v is an eigenvector for B.

"Typical": $AB = BA \Rightarrow$ common eigenvectors, common basis in which they are diagonal.
When is typicality guaranteed? What about infinite-dimensional?
Nub of the problem: $V \neq V^*$.

One more piece of structure: Inner product

(v, w) is an inner product if:

$(v,v) \geq 0$ and $(v,v)=0$ only if $v=0$
$(v,w) = \overline{(w,v)}$
$(\lambda v, w) = (v, \bar{\lambda} w) = \lambda(v,w)$ and $(v_1+v_2, w) = (v_1,w)+(v_2,w)$
[$(v, \ )$ is linear in v]

A Hilbert space is a vector space (over $\mathbb{C}$) with an inner product, for which $(v,v) < \infty$.

Example: Square-integrable functions $\left(\int |f(x)|^2 dx < \infty \right)$
$(f,g) = \int f(x)\overline{g(x)}\, dx$. "$\mathcal{L}^2$".

One typically writes $\|v\|^2 = (v,v)$.
$(\, , \,)$ is a ~~map~~ linear map from V into V^*, namely,

for each w in V, $v \to (v,w)$ is in V^*.

In a Hilbert space, one can find basis sets - in the sense that

$$v = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k v_k \quad \text{means} \quad \lim_{n\to\infty} \|v - \sum_{k=1}^{n} \alpha_k v_k\|^2 = 0$$

$(v,w) = 0 \Leftrightarrow$ "v and w are orthogonal"

4

Self-adjoint operators

<u>Def</u> A is self-adjoint if, for all v, w, $(Av, w) = (v, Aw)$

Example. In $\mathcal{L}^2$: Say $(Af)(t) = \int f(u) A(u,t)\, du$

When is A self-adjoint?

$$(Af, g) = \int\left(\int f(u) A(u,t)\, du\right) \overline{g(t)}\, dt = \int f(u) \overline{g(t)} A(u,t)\, dt\, du$$

$$(f, Ag) = \int f(t) \left(\overline{\int g(u) A(u,t)\, du}\right) dt = \int f(t)\, \overline{g(u)}\, \overline{A(u,t)}\, dt\, du$$

So need $A(u,t) = \overline{A(t,u)}$.

Self-adjoint operators have <u>real</u> <u>eigenvalues</u>. Say $Av = \lambda v$.

Then $\lambda(v,v) = (\lambda v, v) = (Av, v) = (v, Av) = (v, \lambda v) = \bar{\lambda}(v,v)$

Self-adjoint operators have <u>orthogonal</u> <u>eigenvectors</u>, if evals are distinct.

Say $Av = \lambda v$, $Aw = \mu w$.

$\lambda(v,w) = (Av, w) = (v, Aw) = (v, \mu w) = \bar{\mu}(v,w) = \mu(v,w)$

So $(v,w) = 0$.

But eigenvectors may not exist.

$$f(t) \to i\frac{df}{dt} \text{ is self-adjoint in } \mathcal{L}^2$$

$$i\frac{df}{dt} = \lambda f \Rightarrow f = e^{-i\lambda t}$$, but this is not square-integrable.

[fine print: modify $\mathcal{L}^2$ to ensure $\frac{df}{dt}$ is in $\mathcal{L}^2$]

Unitary operators

Unitary operators preserve the structure of a Hilbert space, i.e., U is unitary means

$$(Uv, Uw) = (v,w).$$

Eigenvalues of unitary operators must have magnitude 1:

If $Uv = \lambda v$, then $(Uv, Uv) = (\lambda v, \lambda v)$

$$= \lambda\bar{\lambda}(v,v)$$

But $(Uv, Uv) = (v,v)$. So $|\lambda|^2 = \lambda\bar{\lambda} = 1$.

Eigenvectors of distinct eigenvalues are orthogonal:

Say $Uv = \lambda v$, $Uw = \mu w$.

$(v,w) = (Uv, Uw) = (\lambda v, \mu w) = \lambda\bar{\mu}(v,w)$

If $\lambda \neq \mu$, $\lambda\bar{\mu} \neq 1$ so $(v,w) = 0$.

The unitary operators form a group

For $\mathcal{L}^2$, the unitary operator $D_\tau : f(t) \to f(t-\tau)$ is "natural"

① studying input-output systems:

$$s(t) \to \boxed{X} \to r(t)$$

X should be "stationary" ("time-invariant") – namely

$$X D_\tau = D_\tau X$$

② studying signals:
the statistics should be indep of absolute time

i.e., elements of $\mathrm{Hom}(\mathcal{L}^2 \otimes \cdots \otimes \mathcal{L}^2, k)$ that commute $\bar{c}$ D_τ.

So we'll see how D_τ acts in $\mathcal{L}^2$;
This will define a basis that is natural for the above problems

Rather than solve $D_\tau f = \lambda f$ directly,
we will do it in the general context of
a group represented by unitary op's in $\mathrm{Hom}(V, V)$.

21

Self-adjoint op's $\leftrightarrow$ full set of real eigenvalues
$\leftrightarrow$ $A^T = \overline{A}$

Unitary op's $\leftrightarrow$ full set of eigenvalues of magnitude 1
$\leftrightarrow$ $U^{-1} = \overline{U^T}$

$$U = e^{iAt}$$

Special class of SA op's : projections. A projection is a SA operator P s.t. $P^2 = P$. ["Idempotent"]

All eigenvalues of a projection must be 0 or 1.

For if

$$Pv = \lambda v, \text{ then } Pv = P^2 v = P(Pv) = P(\lambda v) = \lambda^2 v$$

so $\lambda = \lambda^2$, $\Rightarrow \lambda = 0$ or 1.

If P is a projection so is $Q = I - P$. $Q^2 = I - 2P + P^2 = I - P$.

For any w,

$$w = Pw + Qw \quad \text{and}$$
$$(Pw, Qw) = 0.$$

$$\left[\ (Pw, Qw) = (Pw, (I-P)w) = (w, (P-P^2)w) = 0\ \right]$$

So Pw is a projection of w <u>into</u> the subspace spanned by the 1-eigenvectors of P, and Qw is a projection of w into the orthogonal complement.