Fourier Analysis - Applications

Linear Systems Theory

$V: V S$ of functions of time

$F$ linear transform from $V$ to $V$ (not commutes with time translation)

Noted basis for $V (k, V)$ will facilitate work in $F$.

Renewal (and other) point processes

Consider the group algebra of time translation.

The non-negative elements with unit total weight can be thought of as probability distributions.

Evolution of the point process in time is best studied in the natural basis for the group algebra.

Describing analysis of signals

$V: V S$ of functions of time

"Eigenvectors" are "harmonics" are members of $\text{Hom}(V, V)$, $\text{Hom}(V \otimes V, V)$, ...

Select those that commute with time translation.

NB: power spectrum, cross-spectrum, etc.

*Significant issues in estimation from data.*

Non Linear Systems Theory:

$V: V S$ of systems
Examples:
- Electric act: apply voltage (or current) to one pair of terminals, measure current (or voltage) at another pair.
- Circuit: R, C, L

Physical:
- Vestibulo-ocular reflex: note head, eyes move in compensation
- Volume hemodynamics: apply volume (out), urine output adjusts
- Muscle + tendon: apply force, measure displacement
- Membrane: apply voltage (or current) at one location, measure current (or voltage) at another
- Transducer: light to voltage

None of these are exact. Always threshold limits, spending ranges. So the theory can also be extended.

Typical goals: (i) characterize F succinctly
   (ii) find model for F
   (iii) part of F is known, deduce the rest from input, output.
Does the formulation apply?

Time translation, say in $V$:

$$\text{Say } \mathcal{D}_p(s)(t) = s(t+\alpha).$$

Need $(s, s_e) = (D s_1, D s_2).$

This $\Leftrightarrow \int s(t) s_e(t) dt = \int s(t+\alpha) s_e(t+\alpha) dt.$

Just a change of variables, but what about $-$ ?

**F assumed linear in $V$.** Then,

$$\begin{align*}
[f(s_1, s_2)](t) &= [f(s_1)](t) + [f(s_2)](t) \\
[f(s)](t) &= 2[f(s_0)](t)
\end{align*}$$

So, for a complex input $s(t) = a(t) + i b(t)$, one can define

$$[f(s)](t) = [f(a)](t) + i [f(b)](t).$$

The representation of time-translation in $V$ by $\mathcal{D}_p$

is the regular representation.

(i) It contains all irreducible representations [call them $L_\omega$].

(ii) Each copy of the representation $L_\omega$ = dimension of $L_\omega$

(iii) Since $F$ commutes with $\mathcal{D}_p$ and $L_\omega$, it preserves

each subspace in which $\mathcal{D}_p$ acts like $L_\omega$. 

Since time translations are abelian (commutative), all irreducible reps are one-dimensional.

Four implementations:

(a) Time is cyclic and discretely sampled \( \mathbb{N} \) points
\[ G = \mathbb{Z}_N = \{0, 1, \ldots, N-1\} \] under addition (mod \( N \))
\[ \text{time: } j, \text{ discrete freq: } k, \text{ discrete} \]

Periodic signals, discretely sampled.
For each \( k \in \{0, 1, \ldots, N-1\} \), time:
\[ s(j) = s_0, s_1, \ldots, s_{N-1} \] \[ \frac{s_k}{N} = \frac{1}{N} \sum_{j=0}^{N-1} e^{-2\pi i j k/N} \]

(b) Time is cyclic, not discretely sampled.
(a) with \( \frac{j}{N} \rightarrow t \) \[ \text{freq: } k, \text{ discrete} \]
\[ s(t) \text{ on } [0, 1), \text{ periodic on } (-\infty, \infty) \]
\[ s_k = \int_0^1 e^{2\pi i j t} s(t) dt \]
(c) Time not cyclic, not discretely sampled

Make the period progressively longer.

\[ P = 1 \]

\[ P = 2 \]

(d) All \( \omega = \frac{2\pi k}{P} \), \( k + P \) both grow.

For each \( \omega \) (\( \in \mathbb{R} \))

\[ L(\omega)(t) = e^{i\omega t} \]

\[ \hat{S}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} s(t) \, dt \]

(d) Time not cyclic but discretely sampled

Discrete \( \omega \), use \( \omega \Delta t = \frac{\pi}{n} \).

\[ \hat{S}(\omega) = \sum_{j=-\infty}^{\infty} e^{-i\omega j} \tilde{s}_j \]

Note \( \hat{S}(\omega) = \hat{s}(\omega + 2\pi n) \).

* Discretization is bad for \( \omega \approx \frac{1}{2} \) (near \( \text{sampling frequency} \)).

* Nyquist: keep \( \omega < \frac{1}{2} \).
Relationship of $L^2$ functions to "familiar" properties of FT/FS

Recall $X(k) = \sum e^{2\pi ik \cdot t}$, or, $e^{2\pi i k t}$, $m \cdot e^{-2\pi i k t}$.

Fourier inversion = projection onto the subspace corresponding to each $k$.

\[(a) \quad \hat{s}_k = \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i k n} \hat{s}_n \quad \hat{s}_n = \frac{1}{N} \sum_{k=-\infty}^{\infty} e^{2\pi i k n} \hat{s}_k \]

\[(b) \quad \hat{s}_k = \int_{-\infty}^{\infty} e^{-2\pi i k \omega} \hat{s}(\omega) \, d\omega \quad s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{2\pi i \omega t} \hat{s}(\omega) \, d\omega \]

\[(c) \quad \hat{s}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} s(t) \, dt \quad s(t) = \int_{-\infty}^{\infty} e^{i\omega t} \hat{s}(\omega) \, d\omega \]

Fourier synthesis = consequence of characteristic basis functions

Preserved $\|s\|_1$ - the norm is an orthonormal transform, so inner products must be preserved.

\[(a) \quad |\hat{s}_k| = \frac{1}{\sqrt{N}} \|s_k\|_1 \quad \|s_k\|_1 = \sum_{n=0}^{N-1} |s_n| \]

\[(b) \quad \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \int_{-\infty}^{\infty} \hat{s}(\omega)^2 \, d\omega \]

\[(c) \quad \int_{-\infty}^{\infty} |s(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |\hat{s}(\omega)|^2 \, d\omega \]

\[(d) \quad \sum_{k=-\infty}^{\infty} |\hat{s}_k|^2 = \sum_{k=-\infty}^{\infty} \|s_k\|_1^2 \]
Generalized - also been built on larger training domain bases, now referred

(a) \[ \sum_{j=0}^{p} \hat{p}_{j} = \sum_{k=0}^{q} \hat{p}_{k} \]

(b) \[ \sum_{j=0}^{p} (t_{j} q_{j}) = \sum_{k=0}^{q} \hat{p}_{k} \]

(c) \[ \sum_{j=0}^{p} p_{j} q_{j} = \frac{1}{t} \sum_{j=0}^{p} \hat{p}_{j} \]

(d) \[ \sum_{j=0}^{p} q_{j} = \frac{1}{t} \sum_{j=0}^{p} \hat{p}_{j} \]

Consolidation

Consolidation is multiplication in the group algebra, the group algebra has a separate place for each width.

(a) \[ S_{j} = \sum_{j=0}^{p} p_{j-1} q_{j} \]

(b) \[ S(t) = \sum_{j=0}^{p} (t_{j} q_{j}) \]

(c) \[ S(t) = \sum_{j=0}^{p} (t_{j} q_{j}) \]

(d) \[ S(t) = \sum_{j=0}^{p} p_{j-1} q_{j} \]
The linear operator $F$ acts on the subspace corresponding to $L^\infty$, with character $e^{iwt}$, by scalar multiplication.

What is that scalar? (or what does it mean)

Call it $F(w)$.

We know that $s(t) = e^{iwt}$, then $r(t) = F(w) e^{iwt}$

Interpret in terms of real signals.

$s(t) = e^{iwt} = \cos(\omega t) + i\sin(\omega t)$

$w \cdot F(w) = A(w) e^{i\phi(w)} \Rightarrow r(t) = A(w) e^{i(\omega t + \phi(w))}$

$A(w)$ is a scalar (or complex) of amplitude $|F(w)|$

and phase shift $\phi(w)$.

$F(w)$ is the "transfer function"
The transfer function characterizes \( F \) completely.

\[
s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \hat{F}(\omega) \, d\omega
\]

The continuous superposition of cosines \( e^{j\omega t} \) is

Response to \( s(t) \) obeys superposition. \( \hat{F}(\omega) e^{j\omega t} \rightarrow \hat{F}(\omega) \hat{F}(\omega) 
\]

\[
r(t) = \int_{-\infty}^{\infty} e^{j\omega t} \hat{F}(\omega) \, d\omega
\]

i.e., \( \hat{F}(\omega) = \hat{F}(\omega) \hat{F}(\omega) \)

Special case: \( s(t) = \delta(t) \). \( \hat{F}(\omega) = \int_{-\infty}^{\infty} e^{j\omega t} \delta(t) \, dt = 1 \)

Response to a delta-function is

\[
F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \hat{F}(\omega) \, d\omega, \text{ where}
\]

\[
\hat{F}(\omega) = \int_{-\infty}^{\infty} e^{j\omega t} F(t) \, dt
\]

\( F(t) \) is the impulse response. From \( \hat{F}(\omega) = \hat{F}(\omega) \hat{F}(\omega) \) and

\[
r(t) = \int_{-\infty}^{\infty} s(t - \tau) F(\tau) \, d\tau.
\]
\[ r(t) = \int_0^\infty s(t - \tau) F(\tau) \, d\tau \]

1. The \( S \) should be replaced by \( \int_0^\infty \), so \( F(\tau) = 0 \) for \( \tau < 0 \).

\( F(\tau) \) is the "influence" of an input \( T \) in the past.

If \( F(\tau) \neq 0 \) for \( \tau < 0 \), \( T \) is influenced by the future.

Moreover,

2. We could have started with the impulse response.

Any signal \( s(t) \) is a superposition of time-shifted scaled impulses.

\[ s(t) = \int_0^\infty s(\tau) \delta(t - \tau) \, d\tau \]

Knowing the response to an impulse \( \delta(t) \) and \( s(t) \) becomes knowing response to all time-shifted impulses, viz.

\[ \int_0^\infty s(\tau) F(t - \tau) \, d\tau \] - weighted overlaps

\[ \int_0^\infty s(t - \tau) F(\tau) \, d\tau \] - why not measure in the time domain? Range of input, \( S/N \).
Combining linear systems in parallel

\[
\hat{F}(\omega) = \hat{F}(\omega) + \hat{G}(\omega)
\]

\[
\hat{G}(\omega) = \hat{G}(\omega) + \hat{H}(\omega)
\]

\[
\hat{H}(\omega) = \hat{F}(\omega) + \hat{G}(\omega)
\]

Also, \( H(t) = F(t) + G(t) \).

Combining linear systems in series

\[
\hat{F}(\omega) = \hat{F}(\omega) \hat{G}(\omega)
\]

\[
\hat{G}(\omega) = \hat{G}(\omega) \hat{H}(\omega)
\]

So, \( H(\omega) = \hat{F}(\omega) \hat{G}(\omega) \hat{H}(\omega) \).

Also, \( H(t) = F(t) F(t) G(t) \).
\[ G(w) = G(w)F(w) \]

\[ F(w) = \frac{\hat{F}(w)}{1 - \hat{F}(w)G(w)} \]

**Connectivity** ~ algebraic conductance

**Single cell electric circuit (reactor) ~ scale (resistor)**

\[ Q = CV \]
\[ I = C \frac{dV}{dt} \]
\[ V = IR \]

"Expect transfer function to be retained express in \( C \)."