① Fourier Analysis - Applications

Linear Systems Theory

V: VS of functions of time
F: Linear transformations from V to V that commute with
time translation
Natural basis for Hom(V, V) will facilitate work $\bar{c}$ F.

Renewal (and other) point processes
Consider the group algebra of time translation.
The non-negative elements with unit total weight
can be thought of as probability distributions.
Evolution of the point process in time is best studied
in the natural basis for the group algebra.

Descriptive analysis of signals
V: VS of functions of time,
"Measurements" or "moments" are members of
Hom (V, k), Hom (V ⊗ V, k), ...
Select those that commute with time-translation.
NB → power spectrum, cross-spectrum, etc.
* significant issues in ESTIMATION * from data.

Non Linear Systems Theory:
V: VS of SYSTEMS

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LST: $s(t) \rightarrow \boxed{F} \rightarrow r(t)$

Examples: electric ckt, apply voltage (or current) to one pair of terminals, measure current (or voltage) at another pair

(circuit = R, C, L)

physiologic

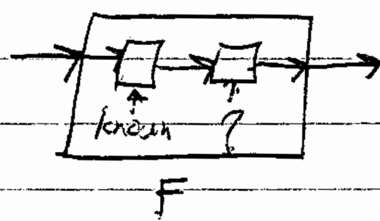
- vestibulo-ocular reflex: rotate head, eyes move ~ compensatorily
- volume homeostasis: apply volume load, urine output adjusts
- muscle & tendon: apply force, measure displacement
- membranes: apply voltage (or current) at one location, measure current (or voltage) at another
- transductions: light to voltage

None of these are exact. Always thresholds, limits, operating ranges. But the theory can also be extended.

Typical goals (i) characterize F succinctly

(ii) test a model for F

(iii) part of F is known, deduce other part from input-output



③

Does the formalism apply?

Time-translation: is it unitary in V?

Say $D_\tau(s)(t) = s(t+\tau)$.

Need $(s_1, s_2) = (Ds_1, Ds_2)$.

This $\Leftrightarrow$ $\int s_1(t)\overline{s_2(t)}dt = \int s_1(t+\tau)\overline{s_2(t+\tau)}dt$.

Just a change of variables, but what about $\overline{}$?

F assumed linear in V. That is,

$$[F(s_1+s_2)](t) = [F(s_1)](t) + [F(s_2)](t)$$

and

$$[F(\lambda s)](t) = \lambda [F(s)(t)]$$

So, for a complex input $s(t) = a(t) + i b(t)$,

one can define

$$[F(s)](t) = [F(a)](t) + i[F(b)](t).$$

The representation of time-translations in V by D_τ is the regular representation.

$\therefore$ (i) H contains all irred. representations [call them L_ω]

(ii) # of copies of the representation L_ω = dimension of L_ω

(iii) Since F commutes with D_τ ($+L_\omega$), it preserves each subspace in which D_τ acts like L_ω

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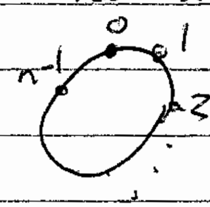
Since time-translations are Abelian (commutative), all irreducible rep's are one-dimensional.

Four "implementations":

(a) Time is cyclic and discretely sampled (N points)

$$G = \mathbb{Z}_N = \{0, 1, \cdots, N-1\} \text{ under addition (mod } N)$$

$\boxed{\text{time: } j, \text{ discrete. freq: } k, \text{ discrete.}}$



Periodic signals, discretely sampled.

For each $k \in \{0, 1, \cdots, N-1\}$ there is an irred rep. of G: $L^{(k)}(j) = e^{\frac{2\pi i}{N} kj}$

$$s(j) = s_0, s_1, \cdots, s_{n-1} \quad ; \quad \hat{s}_k = \frac{1}{N} \sum e^{-\frac{2\pi i}{N} kj} s_j$$
$$= \langle e^{-\frac{2\pi i}{N} kj} s_j \rangle$$

(b) Time is cyclic, not discretely sampled.

(a) with $\frac{j}{N} \rightarrow t$ freq $= k$, discrete

$s(t)$ on $[0, 1)$, or, periodic on $(-\infty, \infty)$

$$\hat{s}_k = \langle e^{-2\pi i kt} s(t) \rangle = \int_0^1 e^{-2\pi i kt} s(t)\,dt$$

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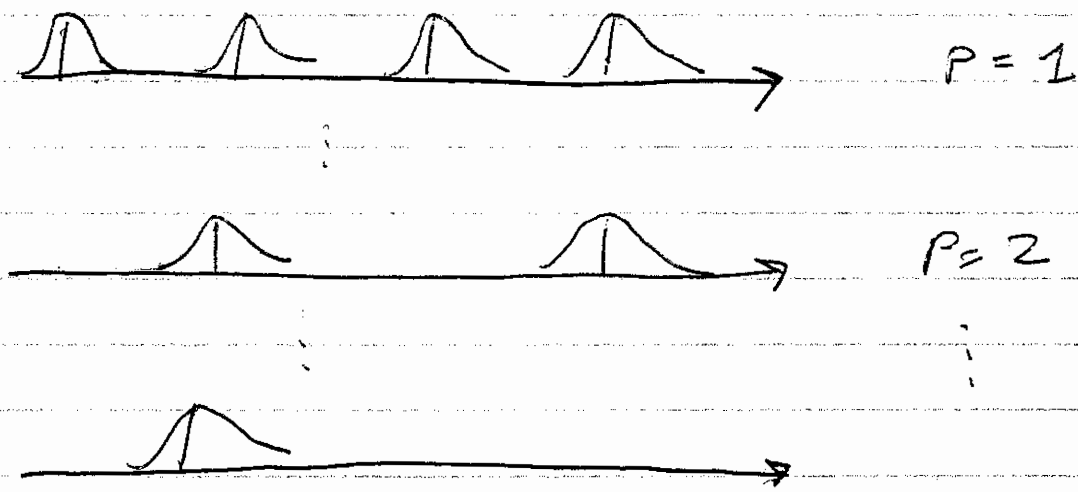
(c) Time not cyclic, not discretely sampled

Make the period progressively longer:



(b) with $\omega = \frac{2\pi k}{P}$, k + P both grow

For each ω ($\in \mathbb{R}$) $\quad L^{(\omega)}(t) = e^{i\omega t}$

$$\hat{s}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t'} s(t)\,dt$$

(d) Time not cyclic but discretely sampled.
Discretize (c). use $\Delta t = 1$.

$$\hat{s}(\omega) = \sum_{j=-\infty}^{\infty} e^{-i\omega j} s_j$$

Note $\hat{s}(\omega) = \hat{s}(\omega + 2\pi)$.

* Discretization is bad for $\omega \approx 1$ $\left(\approx \frac{1}{\text{SAMPLING PERIOD}}\right)$
* Nyquist: keep $\omega << \frac{1}{2}$.

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Relationship of GR formalism to "familiar" properties of FT/FS

Recall $\chi_{L(k)}(j) = e^{\frac{2\pi i}{N}kj}$, or, $e^{2\pi i k t}$, or, $e^{i\omega t}$, or, $e^{i\omega j}$

Fourier inversion = projection onto the subspaces corr to each rep
$\Downarrow$

(a) $\hat{s}_k = \frac{1}{N}\sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N}kj} s_j$ $\qquad s_j = \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}kj}\hat{s}_k$

(b) $\hat{s}_k = \int_0^1 e^{-2\pi i k t} s(t)\,dt$ $\qquad s(t) = \sum_{k=-\infty}^{\infty} e^{2\pi i k t}\hat{s}_k$

(c) $\hat{s}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} s(t)\,dt$ $\qquad s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{s}(\omega)\,d\omega$

(d) $\hat{s}(\omega) = \sum_{j=-\infty}^{\infty} e^{-i\omega j} s_j$ $\qquad s_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega j}\hat{s}(\omega)\,d\omega$

Fourier synthesis = consequence of characters being orthonormal functions

PARSEVAL'S THM - The above is an orthonormal transformation so inner products must be preserved.

(a) $\sum_{j=0}^{N-1}|s_j|^2 = \sum_{k=0}^{N-1}|\hat{s}_k|^2$

(b) $\int_0^1 |s(t)|^2 dt = \sum_{k=-\infty}^{\infty} |\hat{s}_k|^2$

(c) $\int_{-\infty}^{\infty}|s(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{s}(\omega)|^2 d\omega$

(d) $\sum_{j=-\infty}^{\infty}|s_j|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{s}(\omega)|^2 d\omega$

Generalized - also because time-domain basis & frequency-domain bases are orthonormal

(a) $\sum_{j=0}^{N-1} p_j q_j = \sum_{k=0}^{N-1} \hat{p}_k \overline{\hat{q}_k} = \sum_{k=0} \hat{p}_k \hat{q}_{-k}$

(b) $\int_0^1 p(t) q(t) dt = \sum_{k=-\infty}^{\infty} \hat{p}_k \overline{\hat{q}_k} = \sum_{k=-\infty}^{\infty} \hat{p}_k \hat{q}_{-k}$

(c) $\int_{-\infty}^{\infty} p(t) q(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(\omega) \overline{\hat{q}(\omega)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(\omega) \hat{q}(-\omega) d\omega$

(d) $\sum_{j=-\infty}^{\infty} p_j q_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{p}(\omega) \overline{\hat{q}(\omega)} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{p}(\omega) \hat{q}(-\omega) d\omega$

[Last inequality uses, e.g., $\overline{\hat{s}_k} = \sum_{j=0}^{N-1} e^{-\frac{2\pi i k j}{N}} s_j = \hat{s}_{-k}$]

Convolution Thm:

Convolution is multiplication in the group algebra, & the group algebra has a separate piece for each irred. rep.

(a) Say $s_j = \frac{1}{N} \sum_r p_{j-r} q_r$. Then $\hat{s}_k = \hat{p}_k \hat{q}_k$

(b) Say $s(t) = \int_0^1 p(t-\tau) q(\tau) d\tau$. [cyclic p, q] Then $\hat{s}_k = \hat{p}_k \hat{q}_k$

(c) Say $s(t) = \int_{-\infty}^{\infty} p(t-\tau) q(\tau) d\tau$. Then $\hat{s}(\omega) = \hat{p}(\omega) \hat{q}(\omega)$.

(d) Say $s_j = \sum_{r=-\infty}^{\infty} p_{j-r} q_r$. Then $\hat{s}(\omega) = \hat{p}(\omega) \hat{q}(\omega)$.

Specifics related to Linear Systems $[L^{(\omega)}]$

The linear operator F acts on the subspace corresponding to $L^{(\omega)}$, with character $e^{i\omega t}$, by scalar multiplication.

What is that scalar? (+ what does it mean)

Call it $\hat{F}(\omega)$.

$$\xrightarrow{s} \boxed{F} \xrightarrow{r}$$

We know that if $s(t) = e^{i\omega t}$, then $r(t) = \hat{F}(\omega)\, e^{i\omega t}$

Interpret in terms of real signals

$$s(t) = e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

With $\hat{F}(\omega) = A(\omega)\, e^{i\phi(\omega)}$, $r(t) = A(\omega)\, e^{i(\omega t + \phi(\omega))}$

$$r(t) = A(\omega) \cos(\omega t + \phi(\omega)) + i\, A(\omega) \sin(\omega t + \phi(\omega))$$

Output is a sinusoid (or cosinusoid) of amplitude $|\hat{F}(\omega)|$ and phase shift $\phi(\omega)$.

$\hat{F}(\omega)$ is the "transfer function".

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The transfer function characterizes F completely.
Concretely, we can write any

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{s}(\omega) d\omega$$

a continuous superposition of sinusoids $\hat{s}(\omega) e^{i\omega t}$

Response to $s(t)$ obeys superposition $\hat{s}(\omega) e^{i\omega t} \to \hat{s}(\omega)\hat{F}(\omega) e^{i\omega t}$

$$r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[e^{i\omega t} \hat{F}(\omega)\right] \hat{s}(\omega) d\omega$$

i.e., $\boxed{\hat{r}(\omega) = \hat{F}(\omega)\hat{s}(\omega).}$

Special case: $s(t) = \delta(t)$. $\hat{s}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \delta(t) dt = 1$

Response to a delta-function is

$$F(t) = \frac{1}{2\pi} \int e^{i\omega t} \hat{F}(\omega) d\omega \text{ , where}$$

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} F(t) dt$$

$F(t)$ = "impulse response". From $\hat{r}(\omega) = \hat{F}(\omega)\hat{s}(\omega)$ and convolution thm,

$$r(t) = \int_{-\infty}^{\infty} s(t-\tau) F(\tau) \, d\tau.$$

(10)

Comments on

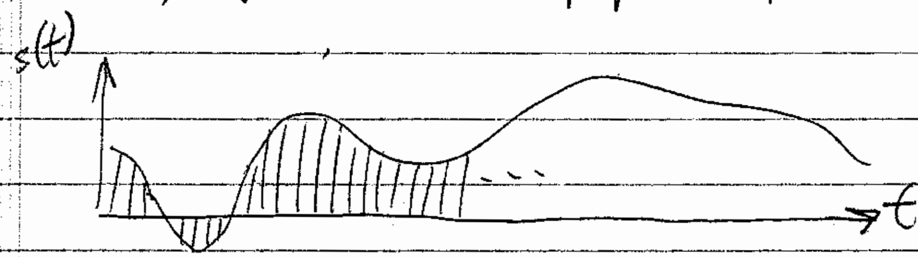
$$r(t) = \int_{-\infty}^{\infty} s(t-\tau)\, F(\tau)\, d\tau$$

1. The $\int$ should be replaced by $\int_0^{\infty}$, i.e., $F(\tau) = 0$ for $\tau < 0$.

$F(\tau)$ is the "influence" of an input τ in the past;
if $F(\tau) \neq 0$ for $\tau < 0$, r is influenced by the future.
More later.

2. We could have started with the impulse response

Any signal $s(t)$ is a superposition of time-shifted, scaled impulses



Since $\left[\, s(t) = \int s(\tau)\, \delta(t-\tau)\, d\tau \,\right]$

Knowing the response to an impulse of unit size at time 0
means knowing responses to all time-shifted impulses, via

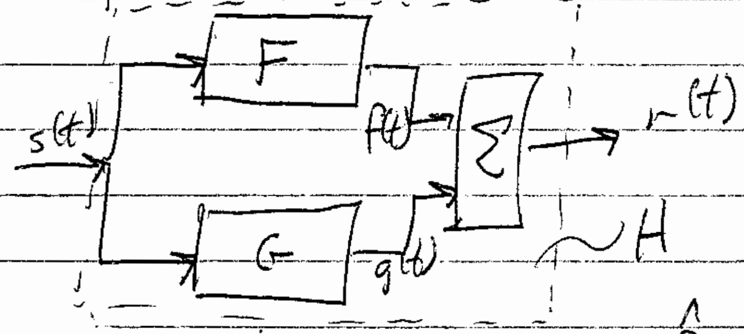
$$\int s(\tau)\, F(t-\tau)\, d\tau \qquad \text{weighted superpos of impulse responses}$$

$$\| $$

$$\int s(t-\tau)\, F(\tau)\, d\tau$$

Why not measure in the time domain? Range of linearity. S/N.

Combining linear systems in PARALLEL

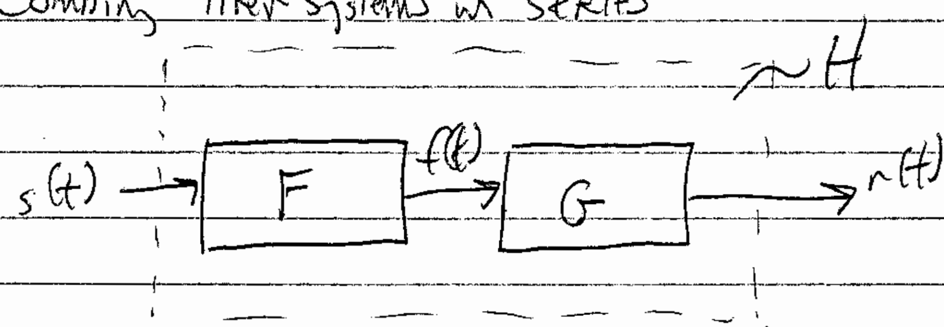


$$\hat{f}(\omega) = \hat{F}(\omega)\,\hat{s}(\omega)$$
$$\hat{g}(\omega) = \hat{G}(\omega)\,\hat{s}(\omega)$$

$$\hat{r}(\omega) = \hat{f}(\omega) + \hat{g}(\omega)$$
$$= \hat{F}(\omega)\hat{s}(\omega) + \hat{G}(\omega)\hat{s}(\omega) = \left[\hat{F}(\omega) + \hat{G}(\omega)\right]\hat{s}(\omega)$$

$$\hat{H}(\omega) = \hat{F}(\omega) + \hat{G}(\omega).$$ Also, $H(t) = F(t) + G(t)$.

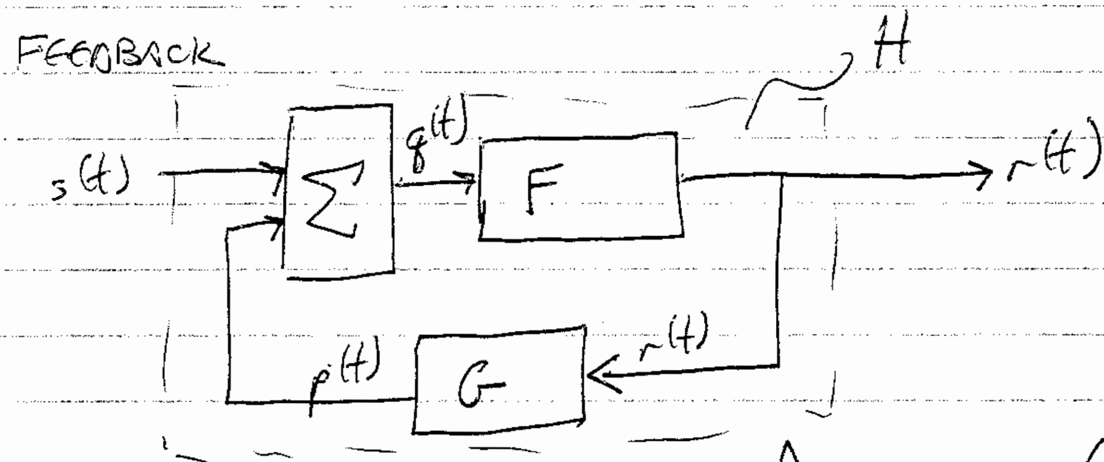
Combining linear systems in SERIES



$\hat{f}(\omega) = \hat{F}(\omega)\hat{s}(\omega)$, $\hat{r}(\omega) = \hat{G}(\omega)\hat{f}(\omega) = \hat{G}(\omega)\hat{F}(\omega)\hat{s}(\omega)$

So $\hat{H}(\omega) = \hat{F}(\omega)\hat{G}(\omega)$. Also, $H(t) = \int F(t-\tau)G(\tau)\,d\tau$

FEEDBACK



$$\hat{q}(\omega) = \hat{p}(\omega) + \hat{s}(\omega), \quad \hat{p}(\omega) = \hat{G}(\omega)\hat{r}(\omega), \quad \hat{r}(\omega) = \hat{F}(\omega)\hat{q}(\omega)$$

$$\hat{r}(\omega) = \hat{F}(\omega)\left(\hat{p}(\omega) + \hat{s}(\omega)\right)$$

$$= \hat{F}(\omega)\left(\hat{G}(\omega)\hat{r}(\omega) + \hat{s}(\omega)\right)$$

$$\hat{r}(\omega)\left[1 - \hat{F}(\omega)\hat{G}(\omega)\right] = \hat{F}(\omega)\hat{s}(\omega)$$

$$\hat{r}(\omega) = \frac{\hat{F}(\omega)}{1 - \hat{F}(\omega)\hat{G}(\omega)}\hat{s}(\omega).$$

$$\hat{H}(\omega) = \frac{\hat{F}(\omega)}{1 - \hat{F}(\omega)\hat{G}(\omega)}$$

Connectivity $\rightleftarrows$ algebraic combination.

Simple ckt elements differentiate (capacitor) or scale (resistor)

$$Q = CV$$

$$I = C\frac{dV}{dt} \qquad V = IR$$

Expect transfer fns to be rational expressions in ω.