Groups, Fields, Vector Spaces

General Themes
- avoid the appearance of accidents
- avoid coordinates (or find "natural" coordinates)
- use "good" model models for data
- smooth transition back to large x infinite
- smooth transition back to sampled - conditions

Overview:
- groups - one kind of element, one operation
- fields - add a second operation
- vector spaces "over" a field
- transformations from one vector space to itself
- groups can be modeled as these transformations
- characterizing these transformations leads to "natural" bases ("coordinates") in the VS
- Fourier theory is a special case of the above

\[ G = \mathbb{Z}_n \text{ (under addition) } \rightarrow \text{ Discrete Fourier Trans.} \]
\[ G = \mathbb{R} \text{ (under addition) } \rightarrow \text{ Fourier Trans.} \]
\[ G = \text{rotations of a circle } \rightarrow \text{ Fourier Series} \]

\[ G = \text{rotation of a sphere } \rightarrow \text{ spheroïd forms.} \]
\[ G = \text{permutations of } n \text{ items} \]
\[ G = \text{translation of } \mathbb{R}^n \]
\[ G = \text{translation + rotation of } \mathbb{R}^n \]
other useful things
Group actions

G1) Associativity: For all a, b, c, \( a \circ (b \circ c) = (a \circ b) \circ c \).

G2) Identity: There is an element \( e \) such that, for all \( a \), \( a \circ e = a \) and \( e \circ a = a \).

G3) Inverses: For all \( a \), there is a corresponding \( a^{-1} \) such that:
\[
\begin{align*}
a \circ a^{-1} &= e, \\
a^{-1} \circ a &= e.
\end{align*}
\]
Not assumed to be commutative ("Abelian")
May be finite or infinite.
May have other properties (e.g., "nearness" - a topology)
(Lie groups)

Examples:
- \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) under addition
- \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \) with \( 0 \) omitted under multiplication
- \( m \times n \) matrices under addition
- \( m \times n \) invertible matrices under mult.

group properties

- rotations of an \( n \)-sphere
- rotations of a regular \( k \)-gon
- reflections of a regular \( k \)-gon
- permutations of a set of objects

Which are commutative? Which are finite? Which have a nontrivial topology?
Basic properties of identities and inverses:

- Only one identity element. If \(e\) and \(f\) were both identities, then
  \[ e \circ f = e \quad \text{since } f \text{ is an identity} \]
  \[ e \circ f = f \]

- Inverse is unique. Say \(a \circ b = e\).
  Then
  \[ a \circ (a \circ b) = a \circ e = e \]
  \[ (a \circ a) \circ b = a \circ b = e \]
  \[ e \circ b = a \]

- No element has a "right" identity. For \(a \circ e = a\),

  \[ a \circ (a \circ f) = a \circ a \]
  \[ (a \circ a) \circ f = a \circ e \]
  \[ f = e \]

- \((a \circ b) \circ b^{-1} = a^{-1} \circ a\)

  \[(a \circ b) \circ (b^{-1} \circ a^{-1}) = ((a \circ b) \circ b^{-1}) \circ a^{-1}
  = (a \circ (b \circ b^{-1})) \circ a^{-1}
  = (a \circ e) \circ a^{-1}
  = a \circ a^{-1}
  = e.\]
Intrinsic properties: the order of an element $a$ is the least (non-zero) integer $n$ for which $a^n = e$. Write $a^n = e$.

i) For finite groups, every element has an order.

Consider $a^0 (= e), a^1, a^2, a^3, \ldots$

Eventually, there must be a repeat.

If $a^h = a^k$ then, ($i \neq j < k$)

(a) $\exists a^h = a^{k-h}$

Order of $a$ must be $\leq k-h$.

b) For finite groups, the order of every element divides the size of the group. (Lagrange's Theorem.)

Let $A = \{ a^0, a^1, a^2, \ldots \}$. Size of $A$ is order of $a$.

$A$ is a subgroup (i.e., a subset of $G$, but also a group).

Consider the subgroup $H$ divides $16$ for any subgroup $H$.

Say $H = \{ e, h, h^2, h^3, \ldots \}$. Write $H = \{ h, h^2, h^3, \ldots \}$. Note $h$ is a "cyclic" (not necessarily a subgroup).

Consider two cosets $Hb = Hc$. They are either identical or disjoint. If not disjoint, say $Hb = Hc$.

Then $b = h^{-1} c$.

$h^o b = h^o \cdot h^{-1} \cdot c$.

So every element in $Hb$ is in $Hc$. And vice versa.
Adhomorphism: \( f : G \rightarrow G \)

Adhomorphism is into \( H \) if it is injective, i.e., \( f \) is 1-1.

\[ f(x) = x \lor (x \lor y) = y \]

Therefore, \( f \) is a homomorphism.

\[ f(1) = 1 \lor 0 = 1 \]

\[ f(0) = 0 \lor 0 = 0 \]

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Examples

\[ \log : \text{a homomorphism from } (\mathbb{R}^+; \text{multiplication}) \]
\[ \text{to } (\mathbb{R}; \text{addition}) \]

\[ \text{id}_n : \mathbb{Z}_n \rightarrow \mathbb{Z}_n \text{ a homomorphism from } (\mathbb{Z}_n; \text{addition}) \]
\[ \text{to } (\mathbb{Z}_n; \text{addition}) \]

\[ x \rightarrow -x \]

\[ \mathbb{Z} \rightarrow \mathbb{E}^\mathbb{Z} \text{ from } (\mathbb{Z}; \text{addition}) \]
\[ \text{to } (\mathbb{E}; \text{sof}; \text{multiplication}) \]

Parity of a permutation from any permutation group is \([-1; 1]\) under multiplication.

Which are onto? Which are isomorphisms? Which are automorphisms?

The kernel of a homomorphism \( \varphi : G \rightarrow H \)
is the set of elements \( g \) for which \( \varphi(g) = e \) in \( H \).

The kernel is a subgroup. Need to show that (a) \( \varphi(e) = e \)
and (b) \( \varphi(g) = e \) then \( \varphi(g^{-1}) = e \).

(a) \( \varphi(e) = \varphi(e \cdot e) = \varphi(e) \cdot \varphi(e) \) so \( \varphi(e) \) is the identity for \( \varphi(e) \) in \( H \), so \( \varphi(e) \) is in \( \text{ker } \varphi \).

(b) \( \varphi(g) \cdot \varphi(g^{-1}) = \varphi(g \cdot g^{-1}) = \varphi(e) = e \)
\[ e = \varphi(g) \cdot \varphi(g^{-1}) \]
Objects playing several roles.

The automorphisms form a group, $G(\mathbb{C})$.

Say $\psi_1$, $\psi_2$ are both automorphisms.

Define $(\psi_1 \circ \psi_2)(g) = \psi_1(\psi_2(g))$.

[We use $\circ$ for the group op in $G(\mathbb{C})$ write $g h$ within $G$]

Need to show $\psi_1 \circ \psi_2$ is an automorphism

$$(\psi_1 \circ \psi_2)(gh) = \psi_1(\psi_2(gh)) = \psi_1(\psi_2(g)\psi_2(h)) = \psi_1(\psi_2(g)) \psi_1(\psi_2(h)) = (\psi_1 \circ \psi_2)(g) \cdot (\psi_1 \circ \psi_2)(h).$$

Since automorphisms by def. are invertible, there is a $\phi^{-1}$
end, with the definition for $\circ$, $\phi \circ \phi^{-1} = e$.

There are special automorphisms - the "inner" automorphisms.

For each group element $a$ m $G$,

let $\phi_a(g) = ag a^{-1}$.

Note $\phi_a(gh) = ag h a^{-1} = a g a^{-1} \cdot h a^{-1} = \phi_a(g) \phi_a(h)$ and $\phi_a(a) = 1_a^-$, so $\phi_a$ is invertible.

A model for "change of coordinates".
Next we have a natural map (a homomorphism)

\[ \text{Adj}: G \rightarrow A(G) \]

\[ \text{Adj}(g) = g^{-1} \]

"adjoint"

What is its kernel?

\[ \text{ker Adj} \]

\[ \text{ker Adj} \] means

\[ g^{-1}g = e \text{ for all } g \in G \]

So, the kernel of the adjoint is the subset of $G$ that commutes with all of $G$ ("the center")

\[ \text{ker Adj} \] is trivial if $G$ is commutative.

\[ \text{ker Adj} \] is nontrivial for non-commutative $G$.

What is the center of $\text{ker Adj}$?
Fields

Field axioms:

1. $\mathbb{K}$: Commutative group under $+$, identity element denoted 0
2. $\mathbb{K} - 0 \triangleleft \mathbb{K}$: $\forall \alpha, \beta \in \mathbb{K}, \alpha \cdot \beta \in \mathbb{K}$, identity element denoted 1

Identity

Relationship $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ are fields, $\mathbb{Z}$ is not (why?)

$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ (interpret mod n)

May be a field

Need to be sure there are multiplicative inverses.

If $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$, then $n$ is prime. Conversely, if $n$ is composite, and $\alpha$ shares a factor $\geq 1$ within, then there is no solution.

This gives us finite fields of size $\mathbb{F}_p$, any prime.

Can also make finite fields of prime $p^k$ (Galois fields), but this is slightly trickier.
R is "ordered" - a notion of size (<).

C is "algebraically closed": a polynomial \( \sum_{k=0}^{n} a_k x^k \)
always has a root \( \lambda \), with \( \sum_{k=0}^{n} a_k \lambda^k = 0 \).
This will help a lot.

C has an automorphism \( \zeta : i \mapsto -i \), "complex conjugate".

Any \( z \in \mathbb{C} \) can be written as
\[
\begin{align*}
z &= a + bi \\
&= a + b i = \bar{z} \\
&= a - b i.
\end{align*}
\]

\( \pm i + 1 \) are distinguished.

1 is a multiple identity \((\pm i)^2 = 1\).

This idea can be abstracted: generalized.

\( \sum_{k=0}^{n} a_k x^k \) does not have a root and does not factor in \( \mathbb{C} \).
Then an inner sum
\[
\sum_{k=0}^{n} \beta_k x^k \]
form a field.
Vector spaces

(\emptyset = \emptyset, unless stated otherwise)

VS axioms:

- Scalars \(a, b, \ldots\) are elements of a field.
- Vectors \(v, w, \ldots\) which can be added.

- Vectors form a commutative group under addition.

- Scalar multiplication: a rule from \(F \times V \rightarrow V\)

  \[ a \cdot (v + w) = a \cdot v + a \cdot w \]

  \[ (a + b) \cdot v = a \cdot v + b \cdot v \]

  \[ a \cdot (b \cdot v) = (ab) \cdot v \]

  Nothing said about dimension, length, angles, coordinates.

A field can be regarded as a VS over itself

(vector addition = field addition)

scalar mult. = field multiplication.

Ordered \(n\)-tuples of field elements form a VS.

Say \(v = (v_1, v_2, \ldots, v_n)\)

\(w = (w_1, w_2, \ldots, w_n)\)

See \(v + w = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)\)

\(a \cdot v = (a \cdot v_1, a \cdot v_2, \ldots, a \cdot v_n).\)
Functions on a set \( S \subseteq \mathbb{R} \) are anything with vectors form \( \mathbf{v} \in V \):

\[
(f_1 + f_2)(s) = f_1(s) + f_2(s) \\
(\lambda f_1)(s) = \lambda (f_1(s))
\]

"Free Vector Space on \( S \)"

Linear independence: \( v_1, \ldots, v_n \) are "linearly independent" if for all nonzero \( \alpha_1, \ldots, \alpha_n \),

\[
\sum_{k=1}^{n} \alpha_k v_k \neq 0.
\]

Basis: If \( v_1, \ldots, v_n \) are linearly independent and \( v \in V \) can be written as \( v = \sum_{k=1}^{n} \alpha_k v_k \), then \( \{v_1, \ldots, v_n\} \) is a basis for \( V \).

Note that if \( v = \sum_{k=1}^{n} \alpha_k v_k = \sum_{k=1}^{n} \beta_k v_k \)

\[
\sum_{k=1}^{n} (\alpha_k - \beta_k) v_k = 0
\]

\( V \) is then said to be \( n \)-dimensional.

[Alternative bases have the same size.]

Defining a basis makes sense even if \( S, v_1, \ldots, v_n \) not finite.
Unless additional structure is added, no basis set has a privileged role.

Putting together VS’s:

\[
V \times W \text{ vs } V \oplus W \text{ under } (v, w) \mapsto (v, w)
\]

and operation

\[
(z_1 + z_2) = (V_1 + V_2, w_1 + w_2)
\]

\[
V + W \text{ vs } \text{Hom}(V, W), \text{ space of all homomorphisms, via } \phi \mapsto \phi.
\]

To define the VS operations:

\[
\phi_1 : V \rightarrow W \quad \text{and} \quad \phi_2 : V \rightarrow W
\]

(\(\phi_1 + \phi_2\))(v) = \(\phi_1(v) + \phi_2(v)\)

(\(\lambda \phi_1\))(v) = \(\lambda \phi_1(v)\)

If we have chains \(v_1, \ldots, v_m\) for \(V\) and \(w_1, \ldots, w_n\) for \(W\), then consider

\[
\phi_{ab} ( \sum_{k=1}^{m} v_k ) = \lambda_{a,b} \cdot w_b.
\]

This is a homomorphism, and the \(\lambda_{a,b}\) form a basis for \(\text{Hom}(V, W)\).
If we were to write \( v = \sum_k w_k \otimes (\cdots) \),

\[
\begin{align*}
\mathbf{w} &= \begin{pmatrix} w_1 \\
& \ddots \\
& & w_k \end{pmatrix} = \begin{pmatrix} \beta_1 \\
& \vdots \\
& & \beta_k \end{pmatrix} \\
\mathbf{\Phi} &= \begin{pmatrix} \mathbf{V} = \mathbf{W} : \text{corr to} \end{pmatrix} \begin{pmatrix} \gamma_1 \\
& \ddots \\
& & \gamma_m \end{pmatrix} = \begin{pmatrix} \beta_1 \\
& \vdots \\
& & \beta_k \end{pmatrix} \\
\mathbf{\Lambda} &= \begin{pmatrix} \delta_{11} \\
& \ddots \\
& & \delta_{nm} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\
& \vdots \\
& & \gamma_m \end{pmatrix} \begin{pmatrix} \beta_1 \\
& \vdots \\
& & \beta_k \end{pmatrix}
\end{align*}
\]

If \( \mathbf{\Phi} = \sum_{a=1}^{n} \sum_{b=1}^{m} \gamma_{ba} \mathbf{F}_{ab} \).

Specific case \( \text{Hom} (V, W) : \mathbf{W} = \mathbf{K} \)

\[
\text{Hom} (V, \mathbf{K}) = \text{"dual" of } V = (V^*)
\]

All linear maps from \( V \) to \( \mathbf{K} \).

Dual \( V \) has same dimension (if \( \dim V \) finite),

But \( V \) vs. \( V^* \) are not the same.

Ex: \( V \) = continuous functions on \( \mathbb{R} \), indistinguishable.

\( V^* \) contains \( f \) so every \( g \) in \( V \), \( f \rightarrow \int_{-\infty}^{\infty} g(x) f(x) \). \( V^* \) also contains \( f \rightarrow f(x_0) \).

\( f(x-x_0) \) is not in \( V \).
Even when $V$ is finite, it is a good idea to keep $V \oplus V^*$ separate.

Lights vs. mechanisms

OT: difference images vs. discriminants

Another special case: $\text{Hom}(V, V)$

[Our special case is to look for mappings from $V$ to $\text{Hom}(V, V)$ that preserve the group structure].

Recall that “order” is an intrinsic property of a group; othm $g$, $q = |g|$.

There are intrinsic properties of $\varphi \in \text{Hom}(V, V)$, determined by traces.

Note: one more way of putting together a VS: “tensored products.”

V, W vector spaces, then $V \oplus W$ is a VS composed of formal sums of elements $v \oplus w$, with the rules:

\[
V \oplus (W_1 + W_2) = (V \oplus W_1) + (V \oplus W_2)
\]

\[
(V_1 + V_2) \oplus W = (V_1 \oplus W) + (V_2 \oplus W)
\]

\[
2(V \oplus W) = (2V) \oplus W = V \oplus (2W)
\]
If \( v_1, \ldots, v_m \) is a basis for \( V \),
\( w_1, \ldots, w_n \) \( \in \) \( W \) then
\( v_1 \otimes w_1, \ldots, v_m \otimes w_n \) \( \in \) \( \text{a basis for } V \otimes W \).

But \( V \otimes W \) is not the same as \( \text{Hom}(V, W) \),
just like \( V \) is not the same as \( V^* \).

[Note: \( V \otimes W \) is naturally \( \cong \) to \( \text{Hom}(V^*, W) \)]

Consider \( V \otimes^2 = V \otimes V \).

For \( u = \sum v_k \otimes v_k \), we can define \( \sigma(u) = \sum v_k \otimes V_k \).

Also, \( \sigma \) can define \( s(u) = \frac{1}{2} (u + \sigma(u)) \) "symmetrize" \( a(u) = \frac{1}{2} (u - \sigma(u)) \) "antisymmetrize" \( s(a(u)) = a(s(u)) = 0 \)
\( s(\sigma(u)) = 0 \) all follow from \( \sigma(\sigma(u)) = u \).

\( a(s(u)) = s(a(u)) \)

P.q. \( s(\sigma(u)) = s(\frac{1}{2}(u - \sigma(u))) \)
\( = \frac{1}{4} [s(u) - s(\sigma(u))] \)
\( = \frac{1}{4} [\frac{1}{2}(u + \sigma(u)) - \frac{1}{2}(\sigma(u) + \sigma^2(u))] \)
\( = 0 \)

For every pair \( v^{(2)}, v^{(3)} \) of vectors \( v \in V \), we have
\( \text{vector } s(v^{(2)}) \otimes v^{(3)} \) in \( s(V \otimes^2) \)
and \( a(v^{(2)}) \otimes v^{(3)} \) in \( a(V \otimes^2) \).
This generalizes to $V \otimes V \otimes \cdots \otimes V$
$r$ times.

$r = 2$:

$s(u) = \frac{1}{2} \left( V^{(1)} \otimes V^{(2)} + V^{(2)} \otimes V^{(1)} \right)$
$a(u) = \frac{1}{2} \left( V^{(1)} \otimes V^{(2)} - V^{(2)} \otimes V^{(1)} \right)$

General: $u = V^{(1)} \otimes V^{(2)} \otimes \cdots \otimes V^{(r)}$

Set $s_1(u)$ = $\frac{1}{r!} \sum_{\text{permutations}} \delta_{\gamma}(u)$
where $\delta_{\gamma}(u) = V^{\gamma(1)} \otimes \cdots \otimes V^{\gamma(r)}$

For example,

For $\pi = (15)(23)(12)(17)$,

\[
\begin{align*}
1 &\rightarrow 5 \\
2 &\rightarrow 3 \\
3 &\rightarrow 1 \\
4 &\rightarrow 7 \\
5 &\rightarrow 1 \\
6 &\rightarrow 2 \\
7 &\rightarrow 2
\end{align*}
\]

For $\pi = (3526)$,

\[
\begin{align*}
1 &\rightarrow 7 \\
2 &\rightarrow 6 \\
3 &\rightarrow 5 \\
4 &\rightarrow 4 \\
5 &\rightarrow 2 \\
6 &\rightarrow 7 \\
7 &\rightarrow 3
\end{align*}
\]

Every permutation has an "even" or "parity" of $r$ flips.

For example,

Upper: 1 \rightarrow 5
2 \rightarrow 3
3 \rightarrow 1
4 \rightarrow 7
5 \rightarrow 1
6 \rightarrow 2
7 \rightarrow 2

For $\pi = (15)(23)(12)(17)$,

\[\text{sign}(\pi) = +1\]

For $\pi = (3526)$,

\[\text{sign}(\pi) = -1\]
\[
\alpha = \sum_{i=1}^{n} \frac{1}{m_i} \delta_i
\]

[Need to show that \(\Sigma_i (a_i) = 1\) is independent \(\Gamma\) is decomposed.] Consider

\[
P = \prod_{i<j} (\theta_i - \theta_j)
\]

Then \(\prod_{i<j} (\theta_i - \theta_{\gamma(i)}) = \pm \prod_{i<j} (\theta_i - \theta_j)\).]

Note that a linear transformation \(L\) in \(\text{Hom} \ (V, V)\)
extends naturally to \(V^\otimes r\), \(s(V^\otimes r)\), \(a(V^\otimes r)\)
\(\sigma\), \(a(\otimes^r (a(V^\otimes 1) \otimes \cdots \otimes V)) = a(L^{(1)} \otimes \cdots \otimes L^{(r)})\).

Now let's do a dimension count of \(a(V^\otimes r)\), assuming \(\dim V = n\).

\(r=1:\) trivial, \(a(V) = V\).

\(r=2:\)
\[
a(V_1 \otimes V_2) = \frac{1}{2} (V_1 \otimes V_2 - V_2 \otimes V_1)
\]

This is 0 if \(n_1 = n_2\). Also,
\[
a(V_1 \otimes V_2) = -a(V_2 \otimes V_1)
\]
so these are not linearly independent.

Conclude \(\dim a(V^\otimes 2) = \frac{1}{2} n(n-1)\).
\[ r \geq 3: \quad a(V_n \circ V_n \circ V_n) = 0 \text{ if} \]
any of \( n_1, n_2, n_3 \) equal. Say \( n_k = n_6 \).

Then the pair-swap \((a, b)\) inverts \( \text{sign}(\tau) \),
but leaves \( a(V_1 \circ \cdots \circ V_n) \) unchanged.

\[ \text{sign}(\tau \cdot (ab)) = \text{sign}(\tau) \cdot \text{sign}(ab) = -\text{sign}(\tau) \]

Reordering \( V_n \circ \cdots \circ V_n \) (3! order) yields the same thing.

\[ \dim a(V^{\otimes 3}) = \frac{1}{3!} n(n-1)(n-2). \]

Similarly \( \dim a(V^{\otimes n}) = \frac{1}{n!} n(n-1) \cdots (n-r+1). \)

Let \( r = n \). \( \dim a(V^{\otimes n}) = 1. \)

All elements in \( a(V^{\otimes n}) \) can be written
in the form \( \langle z \rangle \) for some \( z \in a(V^{\otimes n}) \).

Now given \( L \in \text{Hom}(V,V) \), \( a(L^{\otimes n})(z) \) must be
some scalar \( \varepsilon \cdot z \). This scalar is the
determinant of \( L \).

* We didn’t use a basis (yet considered) so \( \det(L) \) is
basis-independent, i.e., intrinsic.

* If, for any basis \( v_1, \ldots, v_n \), \( L(v_1), \ldots, L(v_n) \) are
linearly dependent then \( \det(L) = 0. \) And conversely.
\( \det (L) \) describes how much
\( a(v_1, \cdots, v_n) \) expands, for any \( v_1, \cdots, v_n \).

\( \det (LM) \) can be roughly \( L \) acting in \( MV \).

\[\det LM \propto \frac{a(LMv_1, \cdots, LMv_n) a(Mv_2, \cdots, Mv_n)}{a(v_1, \cdots, v_n) a(v_2, \cdots, v_n)}\]

so \( \det LM = \det L \cdot \det M \).

Characteristic Equation: \( L \) in \( \text{Hom} (V, V) \)

\[ \lambda \text{ in } k \]

\[ I = \text{the identity in } \text{Hom} (V, V) \]

\( L - \lambda I \) is also in \( \text{Hom} (V, V) \).

\[ \det (L - \lambda I) = \text{"characteristic equation of } L." \]

This is a polynomial in \( \lambda \), of degree \( n \).

If \( \det (L - \lambda I) = 0 \), \( \lambda \) is an \( \lambda \) of \( L \).

Then \( L - \lambda I \) maps some nonzero vector \( v \) to \( 0 \).

\( (L - \lambda I)v = 0 \implies Lv = \lambda v \)

\( \lambda \) is an eigenvalue, \( v \) is its eigenvector.
Plan from here: V: functions of time

Inner product
Self-adjoint operator [applying a filter]
Unitary operator [translation in time]

Unitary representations of a group G: translation in time

Decomposes V - a "natural basis"

Coordinates in the natural basis are the Fourier coefficients, transforms.