Homework ⑥-⑧ GFVS (Answers)

1. a. Yes
   b. No (group op. may not be closed) e.g. $G = \{0,1,2,3,4,5,6\}$ under $+ \pmod 6$. $H = \{0,3\}$, $K = \{0,2,4\}$. $2+3 \notin H \cup K$

2. a. If G is commutative, $hn = nh$, so $hN = Nh$.
   b. With $N = N_1 \cap N_2$, note $Nh = N_1 h \cap N_2 h$
$$= hN_1 \cap hN_2$$
$$= hN.$$

   c. Say $k \in K$. It suffices to show that $gkg^{-1} \in K$ for all G

$$\varphi(gkg^{-1}) = \varphi(g)\varphi(k)\varphi(g^{-1}) \quad \text{since } \varphi \text{ is a homo.}$$
$$= \varphi(g)\varphi(g^{-1}) \quad \text{since } k \in K$$
$$= \varphi(geg^{-1}) \quad \text{since } \varphi \text{ is a homo.}$$
$$= \varphi(e) = e.$$

So $gkg^{-1} \in K$.

   d. Let ψ be an arb. elt in $\mathcal{A}(G)$, and let φ_α be the inner automorphism $\varphi_\alpha(g) = \alpha g \alpha^{-1}$.

$$(\psi\varphi_\alpha\psi^{-1})(g) = \psi(\varphi_\alpha(\psi^{-1}(g)))$$
$$= \psi(\alpha\psi^{-1}(g)\alpha^{-1}) \quad [\text{def of } \varphi_\alpha]$$
$$= \psi(\alpha)\psi(\psi^{-1}(g))\psi(\alpha^{-1})$$
$$= \psi(\alpha)\, g\, \psi(\alpha^{-1})$$
$$= \varphi_{\psi(\alpha)}(g)$$

So $\psi\varphi_\alpha\psi^{-1}$ is an inner automorphism.