Homework ⑪ – ⑬ of GFVS (Answers)

1.a $\varphi_{ab}\left(\lambda \sum_k \alpha_k v_k + \lambda' \sum_k \alpha_k' v_k\right)$

$$= \varphi_{ab}\left(\sum_k (\lambda\alpha_k + \lambda'\alpha_k')v_k\right) = (\lambda\alpha_a + \lambda'\alpha_a')w_b$$

$$= \lambda\varphi_{ab}\left(\sum \alpha_k v_k\right) + \lambda'\varphi_{ab}\left(\sum \alpha_k' v_k\right)$$

etc, so φ_{ab} respects VS operations.

b.(1) To show that the φ_{ab}'s are linearly independent

Say $\sum_{a,b} u_{ab}\varphi_{ab} = 0$ for some nonzero u_{ab}'s.

Then $\left(\sum_{a,b} u_{ab}\varphi_{ab}\right)v_k$ must be 0 for all v_k

$$(u_{ab}\varphi_{ab})v_k = 0, \quad k \neq a$$

$$\sum_{ab}(u_{ab}\varphi_{ab})v_k = \sum_b u_{kb}w_b$$

But w_b's form a basis, so $u_{kb} = 0$.
Since this holds for each choice of k, $u_{ab} = 0$.

b.(2) To see that φ_{ab}'s form a basis.
Say ψ is in $\mathrm{Hom}(V,W)$.
Then let u_{ab} be determined by the coef's in W of $\psi(v_a)$, w.r.t. the basis w_b.

HW ⑪-㉑ / GFVS, ctd. (Answers)

3 Char. eq. is $\det\left[\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]$

$= \det\begin{pmatrix}\cos\theta-\lambda & \sin\theta\\ -\sin\theta & \cos\theta-\lambda\end{pmatrix} = \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta$

$= \lambda^2 - 2\lambda\cos\theta + 1.$

Roots are $\lambda = e^{\pm i\theta}$ [eigenvalues]

Eigenvecs: solve

$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}e^{i\theta}x\\ e^{i\theta}y\end{pmatrix}.$$

$x\cos\theta + y\sin\theta = e^{i\theta}x$

$-x\sin\theta + y\cos\theta = e^{i\theta}y$] homogeneous

Take $x = 1$. Then $y = i$. $\begin{pmatrix}x\\ y\end{pmatrix} = \alpha\begin{pmatrix}1\\ i\end{pmatrix}$ is an eigenvec of $e^{i\theta}$

$e^{-i\theta}$: Similarly $x = 1$, $y = -i$, $\begin{pmatrix}x\\ y\end{pmatrix} = \alpha\begin{pmatrix}1\\ -i\end{pmatrix}$ " " " $e^{-i\theta}$

4 c.e. $= \det\begin{pmatrix}5-\lambda & 0 & 0\\ 0 & 5-\lambda & 0\\ 0 & 0 & 5-\lambda\end{pmatrix} = (5-\lambda)^3$. Eigenvals are $\{5\}$.

Since the mtx is the (identity)·5

all vecs are eigenvectors.

5 c.e. $= (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)$; Eigenvals $\{1, 2, 3, 4\}$

Eigenvecs $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

for eigenval $\downarrow$ 1, $\downarrow$ 2, $\downarrow$ 3, $\downarrow$ 4

HW⑪ - ㉑ / eFVs, etc. (Answers)

2. $M = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$, $\det(M - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix}$

$= (3-\lambda)^3$.

Eigenvals = {3}.

Eigenvecs: $3x + y = 3x$, $3y = 3y$ for $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$

So $y = 0$, x = anything. $\begin{pmatrix} x \\ 0 \end{pmatrix}$ are the eigvecs.