II G, F, VS supplement

Examples related to $\otimes$, d

1. Workout 3×3 determinant in coordinates.

L a linear operator, e_1, e_2, e_3 basis vectors

In coordinates e_1, e_2, e_3, can write $Le_j = \sum_{k=1}^{3} L_{jk} e_k$

and this can be used to calculate Lv for $v = v_1 e_1 + v_2 e_2 + v_3 e_3$

$$\overset{\otimes 2}{L}(e_1 \otimes e_3) = Le_1 \otimes Le_3 = \left(\Sigma L_{1k} e_k \right)$$

$$= \left(\sum_{k_1=1}^{3} L_{1k_1} e_{k_1} \right) \otimes \sum_{k_2=1}^{3} L_{3k_2} e_{k_2} \Big)$$

$$\begin{aligned} &= L_{11}L_{31}(e_1\otimes e_1) + L_{11}L_{32}(e_1\otimes e_2) + L_{11}L_{33}(e_1\otimes e_3) \\ &+ L_{12}L_{31}(e_2\otimes e_1) + L_{12}L_{32}(e_2\otimes e_2) + L_{12}L_{33}(e_2\otimes e_3) \\ &+ L_{13}L_{31}(e_3\otimes e_1) + L_{13}L_{32}(e_3\otimes e_2) + L_{13}L_{33}(e_3\otimes e_3) \end{aligned}$$

So if $L = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}$ acting on $\begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}$

$L \otimes L =$

$$
\left(\begin{array}{ccc|ccc|ccc}
L_{11}L_{11} & L_{11}L_{12} & L_{11}L_{13} & L_{12}L_{11} & L_{12}L_{12} & L_{12}L_{13} & L_{13}L_{11} & L_{13}L_{12} & L_{13}L_{13} \\
L_{11}L_{21} & L_{11}L_{22} & L_{11}L_{23} & L_{12}L_{21} & L_{12}L_{22} & L_{12}L_{23} & L_{13}L_{21} & L_{13}L_{22} & L_{13}L_{23} \\
L_{11}L_{31} & L_{11}L_{32} & L_{11}L_{33} & L_{12}L_{31} & L_{12}L_{32} & L_{12}L_{33} & L_{13}L_{31} & L_{13}L_{32} & L_{13}L_{33} \\
\hline
L_{21}L_{11} & L_{21}L_{12} & L_{21}L_{13} & & & & & & \\
L_{21}L_{21} & L_{21}L_{22} & L_{21}L_{23} & & \text{"}L_{22}\cdot L\text{"} & & & \text{"}L_{23}\cdot L\text{"} & \\
L_{21}L_{31} & L_{21}L_{32} & L_{21}L_{33} & & & & & & \\
\hline
 & & & & & & & & \\
 & \text{"}L_{31}\cdot L\text{"} & & & \text{"}L_{32}\cdot L\text{"} & & & \text{"}L_{33}\cdot L\text{"} & \\
 & & & & & & & &
\end{array}\right)
$$

acting on $e_1 \otimes e_1$

$e_1 \otimes e_2$

$e_1 \otimes e_3$

$e_2 \otimes e_1$

$e_2 \otimes e_2$

$e_2 \otimes e_3$

$e_3 \otimes e_1$

$e_3 \otimes e_2$

$e_3 \otimes e_3$, collectively $e_{j_1} \otimes e_{j_2}$

So $(L \otimes L)_{j_1 j_2, k_1 k_2} = L_{j_1 k_1} L_{j_2 k_2}$.

This is $L^{\otimes 2}$.

$$(L \otimes L)(e_{j_1} \otimes e_{j_2}) = \sum_{k_1, k_2} L_{j_1 k_1} L_{j_2 k_2} (e_{k_1} \otimes e_{k_2})$$

$9 = 3^2$ terms

3

To calculate $a(L^{\otimes 2})$, look at basis

$$e_{j_1} \wedge e_{j_2} = e_{j_1} \otimes e_{j_2} - e_{j_2} \otimes e_{j_1} \qquad (j_2 \neq j_1).$$

$$L^{\otimes 2}(e_{j_1} \wedge e_{j_2}) = \sum_{k_1, k_2} \underbrace{\left(L_{j_1 k_1} L_{j_2 k_2} - L_{j_2 k_1} L_{j_1 k_2}\right)}_{\text{Antisymmetric under interchange of } k_1 \,\&\, k_2} (e_{k_1} \otimes e_{k_2})$$

$$= \sum_{\substack{k_1 < k_2 \\ (3 \text{ terms})}} \left(L_{j_1 k_1} L_{j_2 k_2} - L_{j_2 k_1} L_{j_1 k_2}\right)(e_{k_1} \wedge e_{k_2})$$

"Similarly" $L^{\otimes 3}$:

$$(L \otimes L \otimes L)(e_{j_1} \otimes e_{j_2} \otimes e_{j_3}) = \sum_{k_1 k_2 k_3} L_{j_1 k_1} L_{j_2 k_2} L_{j_3 k_3} (e_{k_1} \otimes e_{k_2} \otimes e_{k_3})$$

$27 = 3^3$ terms

with

$$e_{j_1} \wedge e_{j_2} \wedge e_{j_3} = e_{j_1} \otimes e_{j_2} \otimes e_{j_3} - e_{j_2} \otimes e_{j_1} \otimes e_{j_3}$$
$$+ e_{j_2} \otimes e_{j_3} \otimes e_{j_1} - e_{j_3} \otimes e_{j_2} \otimes e_{j_1}$$
$$+ e_{j_3} \otimes e_{j_1} \otimes e_{j_2} - e_{j_1} \otimes e_{j_3} \otimes e_{j_2}$$

$$L^{\otimes 3}(e_{j_1} \wedge e_{j_2} \wedge e_{j_3}) = \sum_{k_1, k_2, k_3} \Big(L_{j_1 k_1} L_{j_2 k_2} L_{j_3 k_3} - L_{j_2 k_1} L_{j_1 k_2} L_{j_3 k_3}$$
$$+ L_{j_2 k_1} L_{j_3 k_2} L_{j_1 k_3} - L_{j_3 k_1} L_{j_2 k_2} L_{j_1 k_3}$$
$$+ L_{j_3 k_1} L_{j_1 k_2} L_{j_2 k_3} - L_{j_1 k_1} L_{j_3 k_2} L_{j_2 k_3}\Big)(e_{k_1} \otimes e_{k_2} \otimes e_{k_3})$$

$$= \sum_{k_1 < k_2 < k_3} (\text{same as above}) \cdot (e_{k_1} \wedge e_{k_2} \wedge e_{k_3})$$

Nontriviality of $\otimes$ (other than for determinants)

Consider only the "3-rotations", namely, 3×3 matrices with real entries & unitary.

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}.$$

$RR^T = I \Rightarrow$ orthogonal columns & rows.

$R \otimes R$ acts on a 9-d space spanned by $e_{j_1} \otimes e_{j_2}$.

It preserves the 3-d space $\left[(e_{j_1} \wedge e_{j_2}) \text{ are bases} \right]$

so it also preserves the 6-d space spanned by $s(e_i \otimes e_j)$, $= s(V^{\otimes 2})$.

Consider W = space spanned by $w = \begin{matrix} e_1 \otimes e_1 \\ 0 \\ 0 \\ 0 \\ e_2 \otimes e_2 \\ 0 \\ 0 \\ 0 \\ e_3 \otimes e_3 \end{matrix}$, a subspace of $s(V^{\otimes 2})$.

What is $(R \otimes R)(W)$?

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Suffices to consider $R(w)$, for above w (since W is 1-d).

According to formula on $\boxed{3}$,

$$R\otimes R\,(w) = \sum_{k_1,k_2} \left(L_{1k_1}L_{1k_2} + L_{2k_1}L_{2k_2} + L_{3k_1}L_{3k_2}\right)(e_{k_1}\otimes e_{k_2})$$

But if $k_1 \neq k_2$, this is 0 (columns of L are orthog.)
If $k_1 = k_2$, this is 1 (columns of L are normal)

So $(R\otimes R)(w) = w$.

So, $s(V^{\otimes 2})$ has a preserved 1-d subspace, with basis vector
$(e_1\otimes e_1, 0, 0, 0, e_2\otimes e_2, 0, 0, 0, e_3\otimes e_3)$.

Remaining space is 5-d $\quad s(V^{\otimes 2}) = W \oplus [\text{5-d space}]$
spanned by, e.g.,
$s(e_1\otimes e_2)$, $s(e_1\otimes e_3)$, $s(e_2\otimes e_3)$,

$e_1\otimes e_1 - e_2\otimes e_2$, $\quad e_1\otimes e_1 - e_3\otimes e_3$.

In this basis, a rotation by an angle θ maps to
a matrix N to

$$\begin{pmatrix} 1 & & & & & \\ & \cos\theta & \sin\theta & & & \\ & -\sin\theta & \cos\theta & & & \\ & & & & \cos 2\theta & \sin 2\theta \\ & & & & -\sin 2\theta & \cos 2\theta \end{pmatrix}$$