Group Representations

G a group. V a VS over $\mathbb{C}$

A group representation ~~to~~ ~~is~~ a group homomorphism from G onto unitary operators $\mathrm{Hom}(V,V)$.

That is, $L: g \to L_g$, with $L_g L_h = L_{gh}$.

Example. Rotations of an n-sphere can be represented in in n-dimensional v.s. V (by their "matrices")
[why is this unitary?]
but also in $V \otimes V$, $a(V\otimes V)$, $s(V\otimes V)$, $V\otimes V\otimes V$, etc.

Rotations of a circle $z \to e^{i\varphi}z$ are represented in $k=\mathbb{C}$, by this action, but also in $k=\mathbb{C}$ by the action $z \to e^{im\varphi}z$ for any integer m.

A group always has a trivial rep. $L_g = I$.
More interestingly, a group also has a representation in the free vector space on the group [the "regular" representation]
$f \in$ "free VS on the group" means that $f(s) \in k$ for any $s \in G$.

Define $L_g f(s) = f(g^{-1}s)$. Then need to check that $L_g L_h = L_{gh}$.

$$L_g L_h f = L_g(L_h f) ; \quad L_g(L_h f)(s) = L_h f(g^{-1}s) = f(h^{-1}g^{-1}s)$$

But $(L_{gh} f)(s) = f((gh)^{-1}s) = f(h^{-1}g^{-1}s)$.

△

A *group algebra* is the set of formal sums $\sum \alpha_k g_k$, for α in k, with associativity as needed.

A representation of the group extends to a structure-preserving map of the group algebra into $Hom(V,V)$

$$L(\sum \alpha_k g_k) = \sum \alpha_k L_{g_k}$$

For the group of time-translations, we can identify any $f(t)$ with an element of the group algebra,

$$f(t) \rightarrow \int f(\tau) D_\tau \, d\tau$$

What is $f(t) g(t)$? [in the group algebra]

$$\left(\int f(\tau) D_\tau \, d\tau\right)\left(\int g(\tau') D_{\tau'} \, d\tau'\right)$$

$$= \iint f(\tau) g(\tau') D_\tau D_{\tau'} \, d\tau d\tau'$$

But $D_\tau D_{\tau'} = D_{\tau+\tau'}$. So, with $u = \tau+\tau'$, above is

$$\int \left[\int f(\tau) g(u-\tau) \, d\tau\right] D_u \, du$$

Multiplication in group algebra = convolution.

Holds for more general groups too. E.g., rotational diffusion.

A concrete, interesting example:

G = permutations of $\{1, 2, 3\}$.

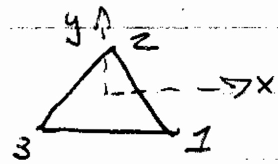
A. One representation. Say $g \in G$ takes 1 to $g(1)$
2 to $g(2)$
3 to $g(3)$.

Then can represent G in a 3-d VS i.e. $\mathbb{E}^3$

$$g \to \begin{pmatrix} 0\text{'s} + 1\text{'s} \end{pmatrix}, \text{ where } (L_g)_{i\,g(i)} = 1, \text{ else } 0$$

e.g., $\begin{matrix} 1\to 2 \\ 2\to 3 \\ 3\to 1 \end{matrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{matrix} 1\to 1 \\ 2\to 3 \\ 3\to 2 \end{matrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. "A"

B. Another representation. Consider a triangle



Every group element is a motion in $\mathbb{E}^2$.

e.g., $\begin{matrix} 1\to 2 \\ 2\to 3 \\ 3\to 1 \end{matrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ where $\theta = \frac{2\pi}{3}$. "B"

Rel. between A + B is not obvious.

Note $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of every A_g, of eval 1.

Let P_v = projection onto v.

Turns out that ⊛ $\mathbb{E}^3 = P(\mathbb{E}^3) \oplus (I - P)(\mathbb{E}^3)$;

⊛ $P(A_g)$ acts trivially in $P(\mathbb{E}^3)$

⊛ $(I-P)(A_g)$ acts "like" B_g in $(I-P)(\mathbb{E}^3)$.

⊛ B cannot be broken down.

④.

For compact ("integrable") and commutative groups, it will turn out that there is a finite or countably infinite set of building-block representations ("irreducible representations"), all of which are finite-dimensional (1-d. if G is commutative)

<u>Equivalent representations:</u> L & M are <u>equivalent</u> if, for some W in $\mathrm{Hom}(V_L, V_M)$ [invertible],

$$L_g = W M_g W^{-1}$$. Note W indep. of g.

<u>Character</u> The character of L at g, $\chi_L(g)$ is

$$\chi_L(g) = tr(L_g).$$

If L & M are equivalent $\chi_L = \chi_M$.

If $g' = \alpha g \alpha^{-1}$, $\chi_L(g') = tr(L_{g'})$
$= tr(L_{\alpha g \alpha^{-1}})$
$= tr(L_\alpha L_g L_{\alpha^{-1}})$
$= tr(L_\alpha L_g (L_\alpha)^{-1})$
$= tr(L_g) = \chi_L(g)$

So χ_L is constant on the conjugate class $\alpha g \alpha^{-1}$ of g.

Composing representations

Say L represents G in $\mathrm{Hom}(V,V)$
Say M " " " $\mathrm{Hom}(W,W)$

$L\oplus M$ represents G in $\mathrm{Hom}(V\oplus W, V\oplus W)$:
defined by $(L\oplus M)_g(v,w) = (L_g(v), M_g(w))$

Note $\chi_{L\oplus M}(g) = \mathrm{tr}\,(L\oplus M)_g = \mathrm{tr}(L_g) + \mathrm{tr}(M_g)$
$= \chi_L(g) + \chi_M(g)$

$$\left(\begin{array}{c|c} L_g & 0 \\ \hline 0 & M_g \end{array}\right)$$

$L\otimes M$ represents G in $\mathrm{Hom}(V\otimes W, V\otimes W)$

defined by $(L\otimes M)_g(v\otimes w) = L_g(v)\otimes M_g(w)$.

$\chi_{L\otimes M}(g) = \chi_L(g)\,\chi_M(g)$

[This follows by noting that the eigenvectors of $L_g\otimes M_g$ are all vecs of form $\varphi\otimes\psi$, where φ is an eigenvec of L_g, ψ an eigenvec of M_g.]

Conjugation less straightforward. Recall for w in V, $w^*\in V^*$ is the map $w^*(v) = (v,w)\in k$.
Now take $\overline{L}_g$ in $\mathrm{Hom}(V^*,V^*)$:
$\overline{L_g}(w^*)(v) = (v, L_g(w))$..

$\chi_{\bar{L}}(g) = \overline{\chi_L(g)}$.

Note $(v, L_g(w)) = (L_g^{-1}(v), w)$ [since L_g is unitary]

but $g \to L_g^{-1}$ is Not a rep. in $\mathrm{Hom}(V,V)$ since

$$L_g^{-1} L_h^{-1} = L_{g^{-1}h^{-1}} = L_{(hg)^{-1}} \neq (L_{gh})^{-1}.$$

The above construction works because order of composition is reversed in going from $\mathrm{Hom}(V,V)$ to $\mathrm{Hom}(V^*, V^*)$.

Irreducible Representation

L is irreducible if it cannot be written as $L_1 \oplus L_2$.

Note that if $L = L_1 \oplus L_2$ and P is a projection onto the space that L_1 operates in, then

$$P L_g = (L_1)_g P.$$

$\boxed{\text{So, } L \text{ irreducible} \Rightarrow P L_g = L_g P \text{ only for } P = kI.}$

It is easy to determine if a representation contains copies of the identity representation: we will create a projection onto the space of preserved vectors, and then count the dimension of that space - i.e., find the trace of that projection.

Say L is a representation (not necessarily irreducible).

Define $P(v) = \frac{1}{\#(G)} \sum_g L_g(v)$ [an average over the group]

1. $L_h P(v) = \frac{1}{\#(G)} \sum_g L_h L_g(v)$ by linearity

$= \frac{1}{\#(G)} \sum_g L_{hg}(v)$ since L is a rep.

$= \frac{1}{\#(G)} \sum_r L_r(v) = P(v)$ by relabelling, $r = hg$.

2. $P^2 v = \frac{1}{\#(G)} L_h P(v) = P(v)$, so $P^2 = P$.

3. P is self-adj, since $(P(v), w) = \frac{1}{\#(G)} \sum_g (L_g(v), w)$

$= \frac{1}{\#(G)} \sum_g (v, L_g^{-1}(w))$

$= \frac{1}{\#(G)} \sum_g (v, L_{g^{-1}}(w))$

[unique inverse] $= \frac{1}{\#(G)} \sum_r (v, L_r(w))$

$= (v, P(w))$

$2 \wedge 3 \Rightarrow P$ is a projection.

$1 \Rightarrow (v = P(v) \Rightarrow L_h v = v)$

$0 \Rightarrow (L_h v = v \Rightarrow P(v) = v)$.

Consequently, # of occurrences of trivial rep. in $L = tr(P)$

$$tr(P) = \frac{1}{\#(G)} \sum_g tr(L_g(v)) = \frac{1}{\#(G)} \sum_g \chi_L(g).$$

8

A "trace formula" for the common irreducible representations of two composite representations

$$\left[\begin{array}{l} L = \underbrace{L_1 \oplus \cdots \oplus L_1}_{l_1 \text{ times}} \oplus \underbrace{L_2 \oplus \cdots \oplus L_2}_{l_2 \text{ times}} \oplus \cdots \oplus \underbrace{L_k \oplus \cdots \oplus L_k}_{l_k \text{ times}} \\ \text{and} \\ M = \underbrace{L_1 \oplus \cdots \oplus L_1}_{m_1 \text{ times}} \oplus \underbrace{L_2 \oplus \cdots \oplus L_2}_{m_2 \text{ times}} \oplus \cdots \oplus \underbrace{L_k \oplus \cdots \oplus L_k}_{m_k \text{ times}} \end{array}\right]$$

This will allow us to

i) determine if L is irreducible
[count common rep's of L with L)

ii) count the # of occurrences of L in the regular representation (p. 4). This will always be >0, so we will have shown that <u>all</u> rep's are contained in the regular representation.

iii) other goodies.

In the setup [ ] above: L_g is in $\mathrm{Hom}(V,V)$.
M_g is in $\mathrm{Hom}(W,W)$.

We construct a representation U in $\mathrm{Hom}(\mathrm{Hom}(V,W), \mathrm{Hom}(V,W))$ and apply the trace formula on p. 7 to U.

To construct U: Need to display a map U_g from $\mathrm{Hom}(V,W)$ to $\mathrm{Hom}(V,W)$.
Say ϕ is in $\mathrm{Hom}(V,W)$
Define $U_g(\phi)$ by $U_g(\phi)(v) = M_g \phi L_g^{-1}(v)$.

U_g is linear

U_g is unitary [follows from unitary nature of L_g, M_g]

U is a representation:

$$U_h(U_g(\phi)) = U_h(M_g \phi L_g^{-1}) = M_h M_g \phi L_g^{-1} L_h^{-1}$$
$$= M_{hg} \phi L_g^{-1} L_h^{-1} = M_{hg} \phi L_{hg}^{-1}$$
$$= U_{hg}(\phi)$$

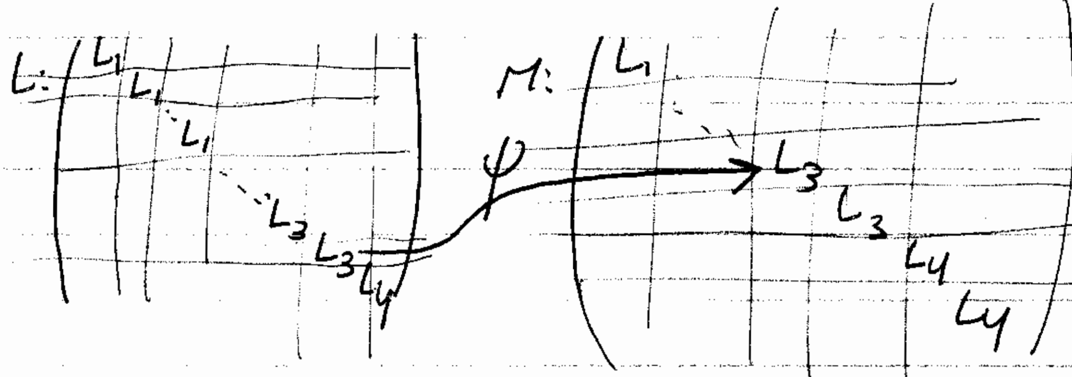
Counting the (dimension of) subspaces of $\mathrm{Hom}(V,W)$ in which U acts like the identity $= \sum l_j m_j$.

Say $U_g \psi = \psi$ for all g. Then

$$M_g \psi L_g^{-1} = \psi, \iff M_g \psi = \psi L_g$$

"Acting in V with L, followed by $\psi: V \to W$, is same as $\psi: V \to W$, + acting with M"

Think of L: M:



10

One such elementary ψ for each pairing of an L_j in V with an L_j in W. ($l_j m_j$ such pairings)

No such ψ's that cut across boundaries because the L_j's are assumed irreducible.

$\sum l_j m_j$ = # of occurrences of trivial rep. in U

$$= \frac{1}{\#(G)} \sum_g \chi_U(g)$$

To calculate χ_U from χ_L + χ_M:

Abstractly, $A^* \otimes B$ can be identified with $\mathrm{Hom}(A,B)$:
For $\alpha^* \in A^*$, $\beta \in B$, need a liner map from A to B associated with $\alpha^* \otimes \beta$ in a natural, (ie, coord-free) linear way:

For v in A, define

$$[\Phi(\alpha^* \otimes \beta)](v) = \alpha^*(v) \cdot \beta$$

Φ exhibits a 1-1 corr betw $A^* \otimes B$ and $\mathrm{Hom}(A,B)$.

U is essentially $\mathrm{Hom}(L, M)$, and can also be thought of as $L^* \otimes M$.

$$\chi_U = \chi_{Hom(L,M)} = \chi_{L^*\otimes M} = \chi_{L^*}\cdot\chi_M \quad [p.\ 5]$$
$$= \overline{\chi_L}\,\chi_M \quad [p.\ 6]$$

Finally, the TRACE FORMULA

$$\sum l_j m_j = \frac{1}{\#(G)}\sum_g \overline{\chi_L(g)}\cdot\chi_M(g)$$

$$= \left\langle \overline{\chi_L(g)}\,\chi_M(g) \right\rangle$$ [average over the group]

For an irreducible rep, this sum must $= 1$
($L = M = $ irred)

For distinct irred reps, ($L \neq M$, both irred) the sum must $= 0$.

∴ Characters are orthonormal functions on the conjugate classes.

Let L be irreducible, M the regular rep, G finite.
Recall: regular rep. acts in space of functions ON the group, or

$$M_g f(s) = f(g^{-1}s).$$

Calculate χ_M. Need a basis for VS of functions on the group. Choose, for each group element α, $e_\alpha(s) = \begin{cases} 0, & s \neq \alpha \\ 1, & s = \alpha \end{cases}$

So, M_g is a permutation matrix in this basis:

$$(M_g e_\alpha)(s) = e_\alpha(g^{-1}s) = \begin{cases} 0, & g^{-1}s \neq \alpha \\ 1, & g^{-1}s = \alpha \\ & (\text{i.e., } s = g\alpha) \end{cases}$$

So $M_g e_\alpha = e_{g\alpha}$.

$tr(M_g) = 0$ unless $g =$ identity
($g\alpha \neq \alpha$ for any α)
$tr(M_{id.}) = \#(G)$ since $M_{id} = I$.

From the trace formula,

$$\frac{1}{\#(G)} \sum_g \overline{\chi_L(g)} \chi_M(g) \quad \text{nonzero only for } g = id.$$
$$= \frac{1}{\#(G)} \overline{\chi_L(id.)} \cdot \#(G)$$

$$= \dim L.$$

\# of occurrences of an irred rep. in the regular rep of G
= its dimension.
Always > 0. So all irred reps occur, and $\sum d_\ell^2 = \#(G)$

If G commutative: \# of conj classes = dim of G

So each $d_\ell' = 1$.

Projections onto a subspace (another applic of the trace formula)

Say U is a composite rep.

$$U = \begin{pmatrix} L_1 & & & & \\ & L_1 & & & \\ & & L_2' & & \\ & & & \ddots & \\ & & & & L_k \end{pmatrix}$$

and L is some irred component

Then $P_L = \frac{\dim L}{\#(G)} \sum_g \overline{\chi_L(g)}\, U_g$ is the projection

onto the subspace on which U acts like L.

[calculation for 1-d rep's L only]: $P_L^2 = P_L$ just like on p. 7

$$\overline{\chi_L(g)}\,\overline{\chi_L(h)} = \overline{\chi_L(gh)} \quad \text{in 1-d case}$$

And in general, $U_h^{-1} P_L U_h = P_L$:

$$U_h^{-1} P_L U_h = \frac{\dim L}{\#(G)} \sum_g \overline{\chi_L(g)}\, U_h^{-1} U_g U_h = \frac{\dim L}{\#(G)} \sum_g \overline{\chi_L(g)}\, U_{h^{-1}gh}$$

$$= \frac{\dim L}{\#(G)} \sum_{g'} \overline{\chi_L(hg'h^{-1})}\, U_{g'}$$

$$= \frac{\dim L}{\#(G)} \sum_{g'} \overline{\chi_L(g')}\, U_{g'} = P_L.$$

So P_L is a projection that preserves irreducible pieces

14

Say U acts like M on a subspace; must have
$P_L = aI$ on that subspace;
can find a by the trace formula

$$\mathrm{tr}(P_L) = \mathrm{tr}\left(\frac{\dim L}{\#(G)} \sum_g \overline{\chi_L(g)}\, \chi_M(g)\right)$$

$$= 0, \quad L \neq M$$
$$\dim L, \quad L = M.$$

So $a = 1$ on the subspace
in which $L = M$.

$$P_L = \frac{\dim L}{\#(G)} \sum_g \overline{\chi_L(g)}\, U_g \text{ is Fourier inversion.}$$