Mathematical Structures in Neuroscience Exam, 2003-2004

Do any three problems. Part A or part B of a problem count as half a problem. Return to my office or email by April 29, 2004.

1. Groups and their representations

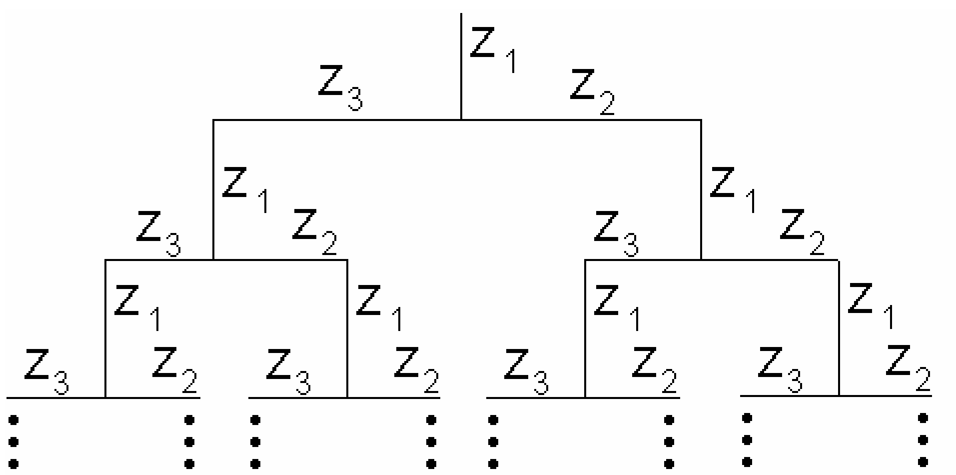
The "dihedral group" D_n is the group of all rotations of a regular n-gon (i.e., rotations by angles of $2pk/n$, along with the mirror-reflections of the n-gon.) That is, D_n has $2n$ elements: the identity, n-1 non-trivial rotations, and n reflections. (D_3 is the 6-element group of rotations and reflections of the triangle, used for numerous examples in class.)

A. Find all the irreducible representations of D_4.

B. Find all the irreducible representations of D_5.

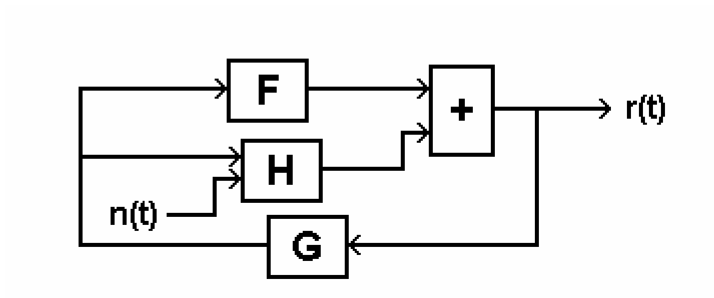
2. Linear systems

Consider a hierarchical "tree" of impedances, as follows. There are 2^n nodes at the nth level. Each node has an impedance Z_1 which enters from above, and two impedances, Z_2 and Z_3, which constitute the outputs to nodes on the subsequent generation. (One might consider this to be the equivalent circuit of a dendritic tree, which branches in a regular pattern.) Assume that this tree is infinite, and at the "bottom" of the tree, all outputs (Z_2 and Z_3) are shorted together ("in the bath"). Find the equivalent impedance of the tree (input at the top, output at the bottom) in terms of Z_1, Z_2, and Z_3. The first three levels of the tree are shown here.



3. Linear systems; power spectra

Consider the system shown below. n(t) represents a "noise" input, whose power spectrum is N(w). What is the power spectrum of r(t)?



4. Gaussians; maximum entropy

A. Let x_1, x_2, and x_3 be Gaussian-distributed random variables with mean zero and variance 1. Assume further that the correlation $\langle x_1 x_2 \rangle = a$, and $\langle x_2 x_3 \rangle = b$. Construct the (multivariate) maximum-entropy distribution for x_1, x_2, and x_3. For this distribution, what is $\langle x_1 x_3 \rangle$?

B. Create Gaussian-distributed random variables x_1, x_2, and x_3 that have mean zero, variance 1, for which $\langle x_1 x_2 \rangle = 0$, $\langle x_2 x_3 \rangle = 0$, but $\langle x_1 x_3 \rangle \neq 0$. Do this for $\langle x_1 x_3 \rangle > 0$ and $\langle x_1 x_3 \rangle < 0$.

5. Mutual Information and the Ideal Observer

Background: Consider a signal p that provides one of k possible symbols, each with probability $1/k$, and a communication channel leading to a response q, also one of k possible symbols. As usual, characterize the communication channel by a matrix r_{ab}, the joint probability that the input p is symbol a, and the response q is the symbol b. Now consider an Ideal Observer (who knows the response q matrix r_{ab}), and whose goal is to guess correctly the lartest fraction of input symbols. The Ideal Observer's strategy, on observing output symbol b, is to guess the input symbol for which r_{ab} is largest.

A. For k=2, assume that the Ideal Observer is correct some fraction f ($1/2<f<1$) of the time. Assume further that each of the k response symbols occurs with probability $1/k$. Find r_{ab} .

B. For k=3, assume that the Ideal Observer is correct some fraction f ($1/3<f<1$) of the time. Assume further that each of the k response symbols occurs with probability $1/k$. Find the communication channel r_{ab} with the *minimum* mutual information that is consistent with this performance. Find the communication channel r_{ab} with the *maximum* mutual information that is consistent with this performance. Explain, qualitatively, what happens to the information in this channel that is not used by the Ideal Observer.