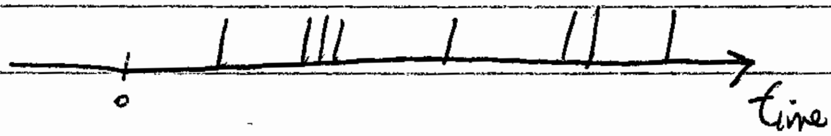
Point Processes

Conceptually – a sequence of (random) events, all identical



More Formally – a (self-consistent) set of probabilities that there there are n_i in each interval $[a_i, b_i)$, for any list of intervals $[a_1, b_1), \cdots [a_N, b_N)$

This is extremely unwieldy. This can all be incorporated into the conditional probability that, given events at times $t_n < t_{n-1} < \cdots < t_1$, and no other events, what is the probability that the next event is at time t? (i.e., is between time t and $t + \Delta t$?)

Can write this as $h(t \mid t_1, \ldots, t_n)$. "the hazard function"

Translation-invariance: $h(t \mid t_1, \ldots, t_n) = h(t-\tau \mid t_1-\tau, \ldots, t_n-\tau)$

Can incorporate a stimulus into the conditional part of h:

$$h(t \mid t_1, \ldots, t_n, s(\cdot))$$

Here we consider translation-invariant point processes, no input.

Renewal process: $h(t \mid t_1, \ldots, t_n) = h(t \mid t_1)$;

time-invariance $\Rightarrow$ $h(t \mid t_1) = h(t - t_1)$

For a renewal process, the hazard function $h(t)$ can be transformed into a "renewal density" $\rho(t)$,

$\rho(t)$ = probability that the first event after an event at time 0 is at time t.

To relate $\rho(t)$ to $h(t)$:
Consider $s(t)$ = "survival function" = probability that there is no event until time t.

$$s(t) = 1 - \int_0^t \rho(t')\,dt'. \Rightarrow \rho(t) = -s'(t).$$

But $\frac{ds}{dt} = -s(t)\,h(t)$, so $\frac{ds}{s} = -h\,dt$

$$s(t) = e^{-\int_0^t h(t')dt'}.$$

$$\text{So } \rho(t) = -s'(t) = h(t)\, e^{-\int_0^t h(t')dt'}$$

What if $h(t) = \lambda$, a constant? Then

$$\rho(t) = \lambda e^{-\lambda t}$$ [Renewal density for a Poisson Process]

Non-renewal: If h depends on prior times then so does ρ.

$$\int_0^\infty \rho(t)\,dt = 1 \text{ or } \int_0^\infty \rho(t \mid t_1, t_2, \ldots, t_n)\,dt = 1.$$

[there is always a next event]

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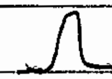
Anything you can do (almost) with real-valued signals can also be done with point processes

Say $x(t) =$  an instance of a p.p.

Smooth this by some $b(t) =$  to get

$y(t) =$ 

for some blurring function $b(t)$.

The blurred signal is $x * b$; $\hat{y}(\omega) = \hat{x}(\omega)\,\hat{b}(\omega)$

But of course $y(t) = x * b(t)$ needs to be sampled anyway.

"Binning" is even worse. An "event" is replaced by  $\updownarrow \frac{1}{\tau}$, but then repositioned in time to line up with bin boundaries. Not linear, not time-invariant.

The "almost" is that we can use any analysis for real-valued signals, as long as it has a limit when applied to $x * b$, as $b(t) \to \delta(t)$.

$\langle y(t) e^{-i\omega t} \rangle$ is one such, since as $b(t) \to \delta(t)$, $\hat{b}(\omega) \to 1$, so $\hat{y}(\omega) \to \hat{x}(\omega)$.

$\left.\begin{array}{l}\text{Impulse responses}\\ \text{Cross-correls}\\ \text{Power spectra}\end{array}\right\}$ OK

Absolute entropy - not ok ($\log \Delta t$).

$\left.\begin{array}{l}\text{Information}\\ \text{Relative entropy}\end{array}\right\}$ OK

Can do things to point processes that you cannot do to real-valued signals:

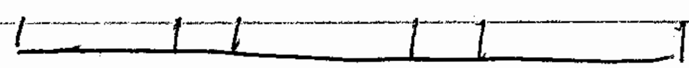
count statistics
interspike interval histograms
serial correlation coefficients

Renewal processes in more detail

Interspike interval distribution $= p(t)$. $\int p(t)\,dt = 1$.

But what is the probability that a time, chosen at random, lies in an interval of length τ to $\tau + \Delta\tau$?
Call this $p_I(\tau)\Delta\tau$

Longer intervals are more likely to be sampled (in proportion to their length)



$$p_I(\tau)\Delta\tau = \frac{\tau\, p(\tau)\,\Delta\tau}{\int_0^\infty t\, p(t)\,dt}.$$

Number of intervals of length τ to $\tau+\Delta\tau$, per unit time, is $\frac{1}{\tau} p_I(\tau)\Delta\tau$.

Number of intervals (all lengths) per unit time = mean rate =

$$\int_0^\infty \frac{1}{\tau} p_I(\tau)\,d\tau = \frac{\int_0^\infty p(\tau)\,d\tau}{\int_0^\infty t\,p(t)\,dt} = \frac{1}{\int_0^\infty t\,p(t)\,dt}.$$

Use $\hat{p}(\omega) = \int_0^\infty e^{-i\omega t} p(t)\,dt$, $p(t) = \int_{-\infty}^\infty e^{i\omega t} \hat{p}(\omega)\,d\omega$

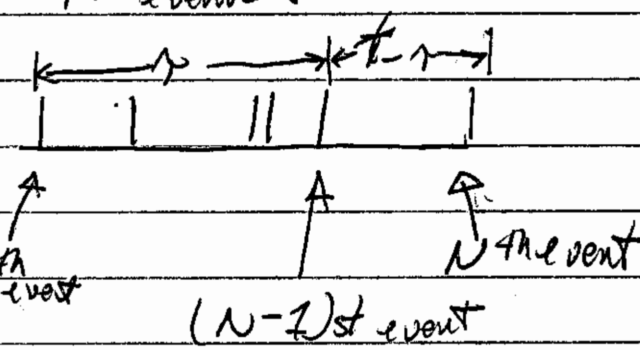
$$\int_0^\infty p(t)\,dt = 1 \Leftrightarrow \hat{p}(0) = 1$$

$$\frac{d}{d\omega}\hat{p}(\omega) = -i\int_0^\infty t e^{-i\omega t} p(t)\,dt, \text{ so, } \hat{p}'(0) = -i\int_0^\infty t\,p(t)\,dt$$

$$\text{mean rate} = \frac{1}{i\hat{p}'(0)}.$$

Distribution of intervals between N events:

Call this $p_N(t)$



$(N-1)$st event can occur at any time in $[0, t)$.

$$p_N(t) = \int_{\tau=0}^{t} p_{N-1}(\tau)\, p(t-\tau)\, d\tau$$

$$\hat{p}_N(\omega) = \hat{p}_{N-1}(\omega)\,\hat{p}(\omega)$$

$$\hat{p}_N(\omega) = \left[\hat{p}(\omega)\right]^N.$$

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Moments of consecutive intervals

M_N = mean of N consecutive intervals:

$$M_N = \int_0^\infty t\, p_N(t)\,dt = i\frac{d}{d\omega}\left[\hat{p}_N(\omega)\right]\Big|_{\omega=0}$$

$$= i\frac{d}{d\omega}\left(\hat{p}(\omega)\right)^N\Big|_{\omega=0}$$

$$= iN[\hat{p}(\omega)]'\left(\hat{p}(\omega)\right)^{N-1}\Big|_{\omega=0}$$

$$= iN\hat{p}(0)' = NM_1$$

[no surprise]

V_N = variance of N consecutive intervals

$$V_N = \int_0^\infty (t-M_N)^2 p_N(t)\,dt = \int_0^\infty t^2 p_N(t)\,dt - M_N^2$$

$$\left(\text{since } \langle (t-M_N)^2\rangle = \langle t^2 - 2M_N t + M_N^2\rangle = \langle t^2\rangle - 2M_N\langle t\rangle + M_N^2 = \langle t^2\rangle - M_N^2\right)$$

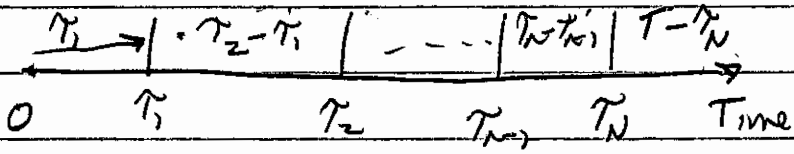
$$\int_0^\infty t^2 p_N(t)\,dt = -\frac{d^2}{d\omega^2}\left(\hat{p}(\omega)\right)^N\Big|_{\omega=0} = -\frac{d}{d\omega}\left(N\hat{p}(\omega)'\left(\hat{p}(\omega)\right)^{N-1}\right)\Big|_{\omega=0}$$

$$= \left[-N(N-1)\left(\hat{p}'(\omega)\right)^2 - N\hat{p}(\omega)''\hat{p}(\omega)^{N-1}\right]_{\omega=0}$$

$$= -N(N-1)\left(\hat{p}'(0)\right)^2 - N\hat{p}''(0)$$

$(\hat{p}'(0))^2 = -M_1^2$ So $V_N = N\left((\tilde{p}'(0))^2 - \tilde{p}''(0)\right)$

Probability that an interval $[0,T)$ contains exactly N events $= K_N(T)$



$$K_N(T) = \int\cdots\int_{0<\tau_1<\tau_2\cdots<\tau_N<T} q(\tau_1)\,p(\tau_2-\tau_1)\,p(\tau_3-\tau_2)\cdots p(\tau_N-\tau_{N-1})\,r(T-\tau_N)\,d\tau_1\cdots d\tau_N$$

where p = renewal density

$q(t)$ = probability that an event occurs at time t, with no events in $[0,t)$, independent of whether an event occurs at time 0

$r(t)$ = probability that, given an event at time 0, next event does not occur for at least t sec.

$$\tilde{K}_N(\omega) = \tilde{q}(\omega)\left[\hat{p}(\omega)\right]^{N-1}\hat{r}(\omega)$$

$$r(t) = \int_t^\infty p(t')\,dt' \; ; \quad r'(t) = -p(t) + \delta(t)$$

hence $i\omega\hat{r}(\omega) = -\hat{p}(\omega) + 1$

$\hat{r}(\omega) = (1-\hat{p}(\omega))/i\omega$

$$q(t) = \frac{\int_t^\infty p(t')\,dt'}{\int_0^\infty t'p(t')\,dt'} = \frac{r(t)}{\int_0^\infty t'p(t')\,dt'} = \frac{r(t)}{i\hat{p}'(0)} = \frac{\hat{p}(\omega)-1}{\omega\hat{p}'(0)}.$$

$N(T)$ = expected # of spikes in an interval of length T.

$$N(T) = \sum_{N=0}^{\infty} N \, K_N(T)$$

$$\hat{N}(\omega) = \sum_{N=0}^{\infty} N \hat{K}_N(\omega) = \sum_{N=0}^{\infty} N \hat{q}(\omega) \hat{p}(\omega)^{N-1} \hat{r}(\omega)$$

$$= \sum_{N=0}^{\infty} N \frac{(\hat{p}(\omega)-1)^2}{-i\omega^2 \hat{p}'(0)} \hat{p}(\omega)^{N-1}$$

$$\sum_{N=0}^{\infty} N \alpha^{N-1} = \begin{array}{l} \alpha^0 + \alpha^1 + \alpha^2 + \alpha^3 + \cdots \\ + \quad \alpha^1 + \alpha^2 + \alpha^3 + \cdots \\ + \quad \alpha^2 + \alpha^3 + \cdots \\ + \quad \alpha^3 + \cdots \end{array}$$

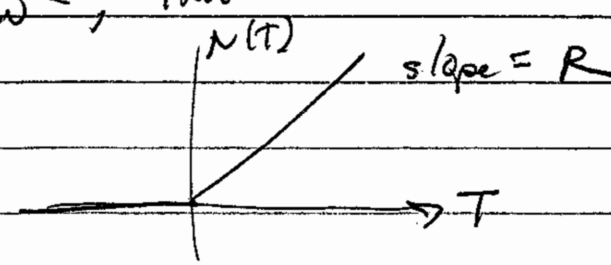
$$= \frac{\alpha^0}{1-\alpha} + \frac{\alpha^1}{1-\alpha} + \frac{\alpha^2}{1-\alpha} + \cdots = \frac{1}{(1-\alpha)^2}$$

So $$\hat{N}(\omega) = \frac{(\hat{p}(\omega)-1)^2}{-i\omega^2 \hat{p}'(0)} \cdot \frac{1}{(\hat{p}(\omega)-1)^2} = \frac{-1}{i\omega^2 \hat{p}'(0)} = \frac{-[\text{mean rate}]}{\omega^2}$$

(p. 5)

If $\hat{N}(\omega) = -\frac{R}{\omega^2}$, then

$N(T) =$



Second deriv of $N(T)$ has F.T. $= R$.

$N'(T) =$