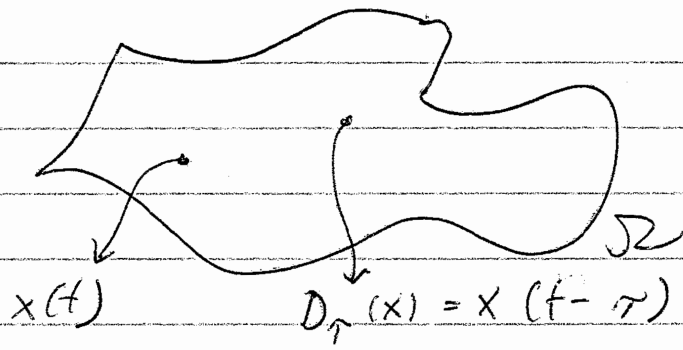
Power Spectra

The general problem: characterize a "noise" $x(t)$.



We typically have access to multiple samples at different start times. So, to get off the ground, assume

a) Stationarity : If $D_\tau(x) = x(t-\tau)$, then $p(D_\tau(x)) = p(x)$

b) Ergodicity $\langle \ \rangle_\Omega = \langle \ \rangle_\tau$

for "all" functionals on $p(x), p(y), \ldots$

c) For suff. large T, $x(t)$ and $x(t+T)$ are independent.

Note that defining $p(x)$ takes some care.

In discrete case, define

$$\begin{aligned} \text{prob.}\big(& x(t_1) \text{ betw } X_1 \text{ \& } X_1 + \Delta X \text{ and} \\ & x(t_2) \text{ '' } X_2 \text{ \& } X_2 + \Delta X, \text{ ''} \\ & \vdots \\ & x(t_k) \text{ '' } X_k \text{ \& } X_k + \Delta X \big) = P(X_1, \ldots, X_k)(\Delta X)^k \end{aligned}$$

and these prob's need to be defined in a self-consistent way, for all $\{X_1, \cdots, X_k\}$, $k = 1, 2, \cdots$

Similarly in continuous time.

We can always take $\langle x(t) \rangle_\Omega = 0$, since

ergodicity $\Rightarrow$ it is constant, & can re-parametrize if $\langle x(t) \rangle_\Omega = \mu \neq 0$.

But of course sample mean may not be 0.

In view of results of last § (maxent + C.L.T.), we will take Ω to be a Gaussian, but parameters of the Gaussian may not be known. Then, extend.

Since $\langle x(t) \rangle_\Omega = 0$, it is not a useful description

what about $C_X(\tau_1, \tau_2) = \langle x(\tau_1) x(\tau_2) \rangle_\Omega$?

Note $C_X(\tau_1, \tau_2) = \langle x(\tau_1 + t) x(\tau_2 + t) \rangle_\Omega$ (any t)

so $C_X(\tau_1 + t, \tau_2 + t) = C_X(\tau_1, \tau_2)$, and we can write

$$C_X(\tau_1, \tau_2) = C_X(\tau_1 - \tau_2), \text{ where } C_X(t) \equiv C_X(t, 0).$$

Also $C_X(\tau_1, \tau_2) = C_X(\tau_2, \tau_1)$, so $C_X(t) = C_X(-t)$.

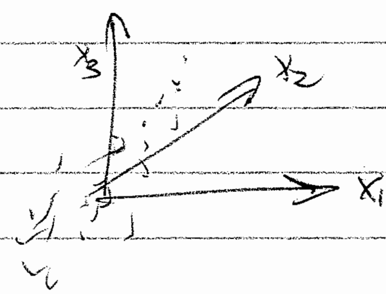
This is the <u>autocovariance</u>. Autocorrel $= C_X(t)/C_X(0)$.

Geometrically:

With $x = (x_1, x_2, \cdots, x_n, \cdots)$,

$C_x(t)$ describes covariance of x_r and x_{r+t}, i.e.,

the shape of the ellipse



But a) translation-in-time symmetry $\Rightarrow$ certain covariances must be equal

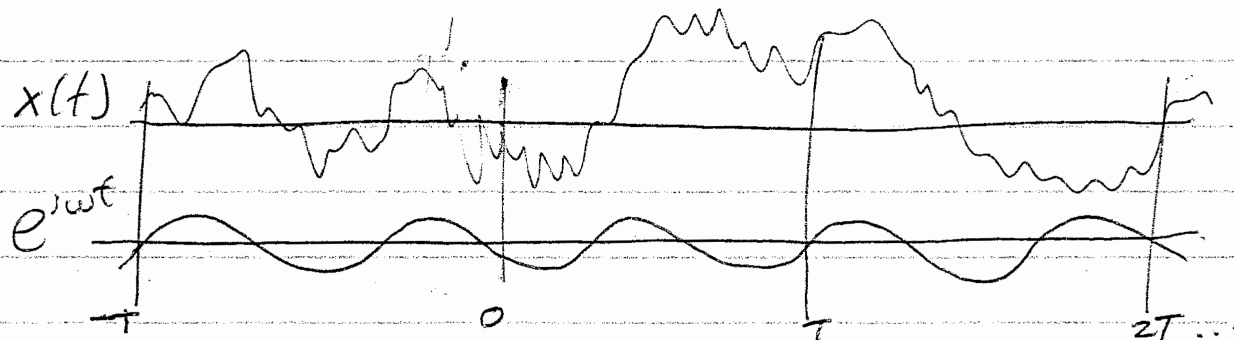
b) Some covariances influence others, i.e.,
if x_3 and x_4 are highly correlated,
then x_4 and x_5 " " "
and x_3 and x_5 must be somewhat correlated

c) Some correlations appear easier to measure
(x_k, x_{k+1}) than others $(x_k, x_{k+\text{large}})$.

So it is not clear how to do estimations –
e.g., how to put error bars on $C_x(t)$

What about trying to characterize $\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$?

This should deal with time-translation. However,
$\tilde{x}(\omega)$ is ill-defined:



$$I = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt = \sum_{N=-\infty}^{\infty} \int_{NT}^{(N+1)T} x(t)\, e^{-i\omega t}\, dt$$

$$= \sum_{N=-\infty}^{\infty} F(x; \omega, T, NT) = \sum_{N=-\infty}^{\infty} F_N$$

F = a "Fourier estimate" at freq. ω, length T, start time NT, of $x(t)$

How are the F's distributed?

$$F(x; \omega, T, T_0) = \int_{T_0}^{T_0+T} x(t)\, e^{-i\omega t}\, dt$$

$u = t + T_0$

$$= \int_0^T x(u - T_0)\, e^{-i\omega(u - T_0)}\, du$$

$$= e^{i\omega T_0} \int_0^T x(u - T_0)\, e^{-i\omega u}\, du$$

$$= F(D_{T_0}(x); \omega, T, 0) \cdot e^{i\omega T_0}$$

But Ω is translation-invariant, so any statistic on x must = that statistic on $D_{T_0}(x)$.

So $F(x;\omega,T,T_0)$ must be distributed like

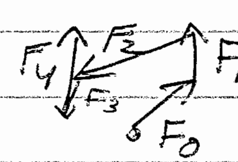
$$F(x;\omega,T,0)\cdot e^{i\omega T_0}.$$

But $F(x;\omega,T,T_0)$ must also be distributed like $F(x;\omega,T,0)$ since when you set "wall clock" time to 0 is arbitrary.

So each $F(x;\omega,T,T_0)$ is distributed circularly-symmetrically in the complex plane.

But also, for T sufficiently large, "most" of the interval of $[NT,(N+1)T]$ is very far from the adjacent intervals, so "most" of each estimate F_N is independent of adjacent, or all other, intervals [condition (c)]

Take $I_{KT} = \sum_{N=0}^{K-1} F_N$, but the F_N's consist of random-walk steps in the plane:



So, $\langle I_{KT}\rangle = 0$ but

$$\langle |I_{KT}|^2\rangle = \langle|\sum_{N=0}^{K-1} F_N|^2\rangle = \sum_{N,M=0}^{K-1}\langle F_N \bar{F}_M\rangle = K\langle|F_0|^2\rangle$$

So $I_{KT} = \int_0^{KT} x(t)\, e^{-i\omega t}\, dt$ has mean 0, and variance proportional to K (fixed T). Proportionality depends on T.

Since T is arbitrary (but large),

I_{KT} must have variance proportional to KT.

Conversely, we expect that

$$\lim_{T\to\infty} \frac{1}{T} \left\langle \left| \int_0^T x(t)\, e^{-i\omega t}\, dt \right|^2 \right\rangle \text{ exists.}$$

This is the <u>power spectrum</u>. Units: $x^2 \cdot T$, or, x^2/Hz

$$P_x(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle \left| \int_0^T x(t)\, e^{-i\omega t}\, dt \right|^2 \right\rangle_\Omega = \lim_{T\to\infty} \frac{1}{T} \left\langle \left| F(x, \omega, T, T_0) \right|^2 \right\rangle_\Omega$$

Because of ergodicity (b), can replace $\langle\ \rangle_\Omega$ by $\langle\ \rangle_{T_0}$.
Because of limited correlation length (c), can replace $\langle\ \rangle_{T_0}$ by

$$P_x(\omega) = \lim_{\substack{T\to\infty \\ N\to\infty}} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{T} \left| F(x, \omega, T, nT + T_0) \right|^2.$$

Note the difficulties with a "plug-in" estimation
- can't make $T \to \infty$
- with data of length L, reciprocity between length of segment T, + max # of segments L/T
- tradeoff betw frequency resolution $\Delta\omega$ and N, since $T \geq 1/\Delta\omega$.

What if there is a deterministic component in $x(t)$, namely,

$$x(t) = x_p(t) + x_N(t)$$

"periodic" "noise"

$$F(x; \omega, T, T_0) = F(x_p, \omega, T, T_0) + F(x_N, \omega, T, T_0).$$

$F(x_N, \omega, T, T_0)$ behaves as above.

But $\langle F(x_p, \omega, T, T_0)\rangle$ typically not 0.

With $T = mP$, P a period of $x_p(t)$, and $\omega = \frac{2\pi h}{P}$

$$\langle F(x_p, \omega, T, T_0)\rangle = \int_{T_0}^{mP+T_0} x_p(t)\, e^{-i\frac{2\pi h}{P}t}\,dt$$

$$= m \int_{T_0}^{P+T_0} x_p(t)\, e^{-i\frac{2\pi h}{P}t}\,dt$$

$$= T \cdot \frac{1}{P} \int_{T_0}^{P+T_0} x_p(t)\, e^{-i\frac{2\pi h}{P}t}\,dt$$

$= T\,(\tilde{x}_p)_h$, i.e., proportional to h^{th} F.C. of x_p.

So, the defining limit of the P.S. does not exist.

However, can write

$$P_{X_n}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \left| F(x,\omega,T,T_0) - \langle F(x,\omega,T,T_0)\rangle \right|^2 \right\rangle_R$$

Since $\langle F(x,\omega,T,T_0)\rangle$ typically not known, it has to be estimated from data. Using standard debiased variance estimate,

$$P_{X_n}(\omega) = \lim_{\substack{T\to\infty \\ N\to\infty}} \frac{1}{N-1}\sum_{n=0}^{N-1} \frac{1}{T}\left| F(x,\omega,T,nT+T_0) - \frac{1}{N}\sum_{n=0}^{N-1} F(x,\omega,T,nT+T_0)\right|^2$$

Why do we not need to consider the more general

$$W_X(\omega_1,\omega_2) = \lim_{T\to\infty} \frac{1}{T}\langle F(x;\omega_1,T,T_0)\,F(x;\omega_2,T,T_0)\rangle,$$

or even

$$W_X(\omega_1,\omega_2,\Delta T_{12}) = \lim_{T\to\infty} \frac{1}{T}\langle F(x,\omega_1,T,T_0)\,F(x;\omega_2,T,T_0+\Delta T_{12})\rangle ?$$

Previous (p. 4-5) argument shows that shifting the start-time T_0 by, say, ΔT and choosing a new sample $D_{\Delta T}(x)$ multiplies the value of $F(x,\omega_1,T,T_0)$ by $e^{i\omega_1\Delta T}$.

Similarly, value of $F(x,\omega_2,T,T_0)$ is multiplied by $e^{i\omega_2\Delta T}$

So the product is multiplied by $e^{i(\omega_1+\omega_2)\Delta T}$

So there will only be a nonzero expectation if, for all ΔT,

$$e^{i(\omega_1+\omega_2)\Delta T} = 0, \text{ i.e., } \omega_1+\omega_2 = 0, \text{ i.e.,}$$

$$W_X(\omega, -\omega) \equiv P_X(\omega) \quad \left[\text{since } F(X,-\omega,T,T_0) = \overline{F(X,\omega,T,T_0)}\right].$$

However, $B_X(\omega_1,\omega_2) \equiv$

$$\lim_{T\to\infty} \frac{1}{T}\left\langle F(X,\omega_1,T,T_0)\, F(X,\omega_2,T,T_0)\, \overline{F(X,\omega_1+\omega_2,T,T_0)} \right\rangle$$

makes sense. "Bispectrum"

What about multichannel data?

Cross-spectrum:

$$C_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle F(X,\omega,T,T_0)\, \overline{F(Y,\omega,T,T_0)} \right\rangle$$

No need to consider two frequencies.

If we can estimate P, we can estimate C_{XY}:

$$\langle (X+Y)\overline{(X+Y)}\rangle = |X|^2 + |Y|^2 + X\bar{Y} + Y\bar{X}$$
$$= |X|^2 + |Y|^2 + 2\,\mathrm{Re}(X\bar{Y})$$
$$\langle (X+iY)\overline{(X+iY)}\rangle = |X|^2 + |Y|^2 + iY\bar{X} - iX\bar{Y}$$
$$= |X|^2 + |Y|^2 + 2\,\mathrm{Im}(X\bar{Y})$$

So,

$$\left.\begin{aligned} \mathrm{Re}\ C_{XY}(\omega) &= \tfrac{1}{2}\left[P_{X+Y}(\omega) - P_X(\omega) - P_Y(\omega)\right] \\ \mathrm{Im}\ C_{XY}(\omega) &= \tfrac{1}{2}\left[P_{X+iY}(\omega) - P_X(\omega) - P_Y(\omega)\right] \end{aligned}\right\} \text{formal}$$

So an estimator for P. will lead to an estimator for C; bias will be easy to calculate.

$$\text{Coherency} = \frac{C_{XY}(\omega)}{\sqrt{P_X(\omega)P_Y(\omega)}} \quad \text{Complex \#}, |\,| \leq 1$$

$$\text{Coherence} = |\text{Coherency}| .$$

Given $P_X(\omega)$, can one calculate a "typical" $x(t)$?

Draw from the max ent distrib specified by $P_X(\omega)$.
Naive approach: generate a signal of length L, starting at $t=0$, sampled $\bar{c}$ resolution Δt.
$F(x, \omega, 0, L)$ is a complex # $\bar{c}$ $\langle |F|^2 \rangle = L\, P_X(\omega)$, symmetrically distributed, i.e.,

$\mathrm{Re}\, F$ has variance $\frac{1}{2} L\, P_X(\omega)$
$\mathrm{Im}\, F$ " " $\frac{1}{2} L\, P_X(\omega)$
$\mathrm{Re}\, F$ + $\mathrm{Im}\, F$ are independently distributed

So, choose $\mathrm{Re}\,F(\omega)$ + $\mathrm{Im}\,F(\omega)$ independently from

$$\frac{1}{\sqrt{2\pi V}}\, e^{-F^2/2V}, \quad V = \frac{1}{2} L P_x(\omega)$$

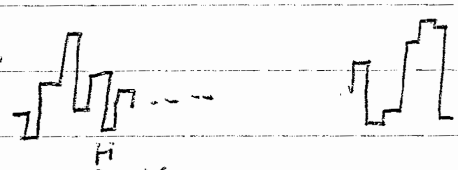
Do this for a sufficient set of ω so that one can write

$$x(t) = \frac{1}{\sqrt{L}} \sum_{\omega = -\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} F(\omega)\, e^{i\omega t}, \quad \text{constant value from } n\Delta t \text{ to } (n+1)\Delta t.$$

in steps of $\frac{1}{L}$ $\quad (N = \frac{L}{\Delta t})$

Note $F(\omega = 0)$ and $F(\omega = \frac{1}{2\Delta t})$ must be real, &
$F(-\omega) = \overline{F(\omega)}$.

The $\frac{1}{\sqrt{L}}$ is to ensure that $\int_0^L x(t)\, e^{-i\omega t}\, dt = \sqrt{L}$

This makes a signal that looks like 
Δt

Problem 1: Its power spectrum, at intermediate freqs, doesn't agree $\bar{c}$ $P_x(\omega)$

2. It is periodic $\bar{c}$ period L.

Solution: use a larger L than you need. then $\Delta\omega$ is smaller.

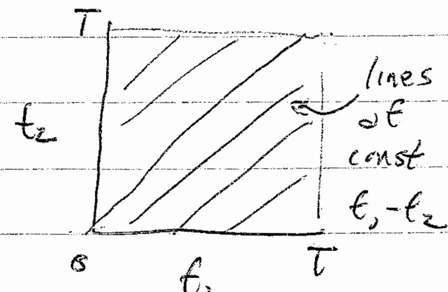
12. Properties of the Power Spectrum

<u>Relation to autocovariance</u>

From $P_X(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle \left| \int_0^T x(t)\, e^{-i\omega t}\, dt \right|^2 \right\rangle$

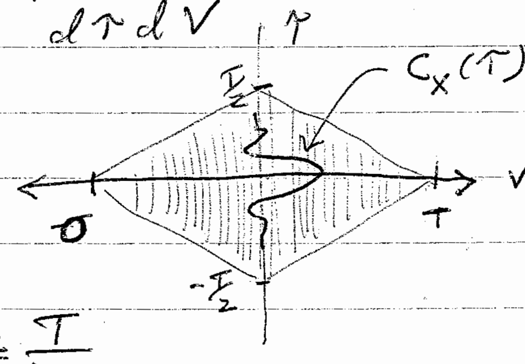
$$= \lim_{T\to\infty} \frac{1}{T} \left\langle \iint_0^T x(t_1)\, x(t_2)\, e^{-i\omega t_1} e^{+i\omega t_2}\, dt_1\, dt_2 \right\rangle$$

$$\tau = t_1 - t_2$$
$$v = \tfrac{1}{2}(t_1 + t_2)$$



$$P_X(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle \iint_{\tau, v} x\left(v + \tfrac{\tau}{2}\right) x\left(v - \tfrac{\tau}{2}\right) e^{-i\omega\tau}\, d\tau\, dv \right\rangle$$

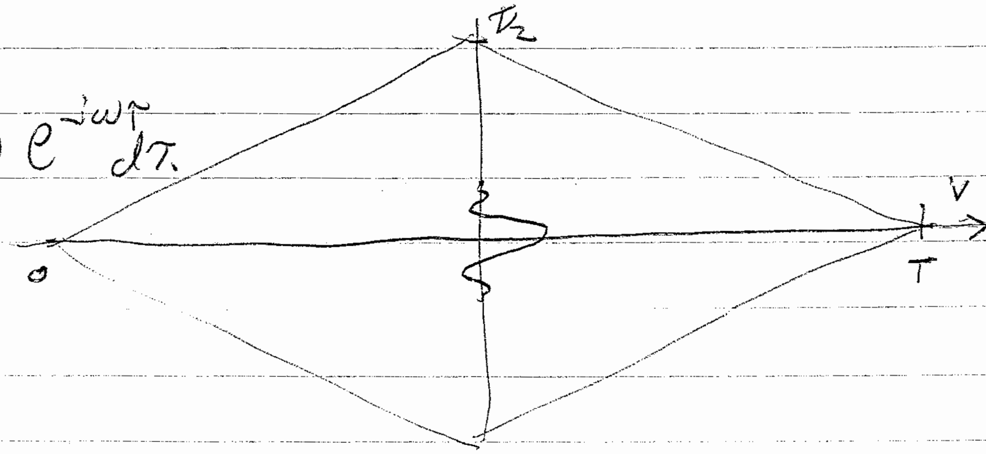
$$= \lim_{T\to\infty} \frac{1}{T} \iint_{\tau, v} C_X(\tau)\, e^{-i\omega\tau}\, d\tau\, dv$$



Once T is large enough so that most of $C_X(\tau)$ has been covered, by, say $|\tau| \le \frac{T}{1-\varepsilon}$ then $\frac{1}{T}\int_0^T dv$ is within $1-\varepsilon$ of 1, where $C(\tau) \neq 0$.

So

$$P_X(\omega) = \int_{-\infty}^{\infty} C_X(\tau)\, e^{-i\omega\tau}\, d\tau.$$



13.

Analogous rel's for bispectrum & cross-spectrum.

(to $\langle x(\tau_1)x(\tau_2)x(-\tau_1-\tau_2)\rangle$) (to cross-covariance)

Superposition property: If $x(t)$ & $y(t)$ are independent,

$$P_{X+Y} = P_X + P_Y$$

[cross-terms in def. of P.S. $\to 0$]

Weaker condition:

If $x(t)$ and $y(t')$ are uncorrelated, $P_{X+Y} = P_X + P_Y$.

Uncorrelated: $\langle x(t)y(t')\rangle = 0$

Independent: $\langle A(x)B(y)\rangle = \langle A(x)\rangle\langle B(y)\rangle$

In either circumstance, $C_{XY}(\omega) = 0$.

Rel. to linear filtering:

$$x(t) \longrightarrow \boxed{G} \longrightarrow y(t).$$

$F(\omega, y) = \hat{G}(\omega)F(\omega, X)$.

$$P_Y(\omega) = \lim_{T\to\infty} \frac{1}{T}\langle|F(Y,\omega)|^2\rangle = \lim_{T\to\infty} \frac{1}{T}\langle|\hat{G}(\omega)F(X,\omega)|^2\rangle$$

$$= |\hat{G}(\omega)|^2 P_X(\omega)$$

Implications:

- $$|G(\omega)|^2 = \frac{P_Y(\omega)}{P_X(\omega)}$$

  Note phase of $G(\omega)$ cannot be determined

- Analysis of connectivity

- Can think of any Gaussian noise as a result of

$$W(t) \to \boxed{G} \to P_X(\omega)$$

$$P_W(\omega) = 1$$

provided $|G(\omega)| = \sqrt{P_X(\omega)}$

Though it may not be clear if phases can be assigned to $G(\omega)$ so that $\int G(\omega) e^{i\omega t} dt$ is causal & finite.