Power Spectra III - Multitaper Estimates

$$P_X(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \left| \int_0^T x(t)\, e^{-i\omega t}\, dt \right|^2 \right\rangle = \lim_{T\to\infty} \frac{1}{T} \left\langle |F(x;\omega,T,T_0)|^2 \right\rangle$$

where $F(x;\omega,T,T_0) = \int_0^T x(t)\, e^{-i\omega t}\, dt$

We can't calculate $P_X(\omega)$ from $\langle |F(x;\omega,T,T_0)|^2 \rangle$, but we can calculate $\langle |F|^2 \rangle$ from P.

More generally, let

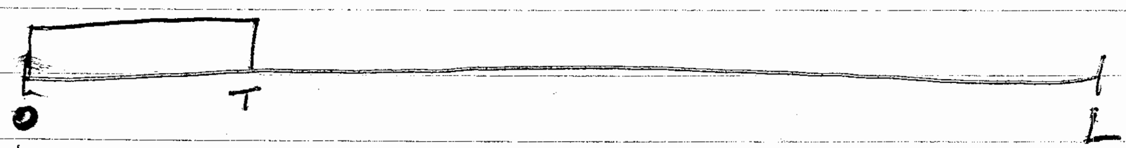
$$F_W(x,\omega) = \int_{-\infty}^{\infty} x(t)\, W(t)\, e^{-i\omega t}\, dt, \text{ where}$$

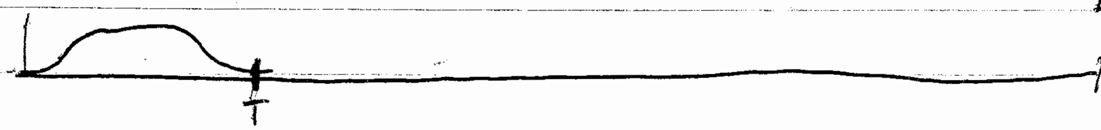
$W(t)$ is a "window", a.k.a. "taper"

Calculate $\langle |F_W(x,\omega)|^2 \rangle$ from $P_X(\omega)$.

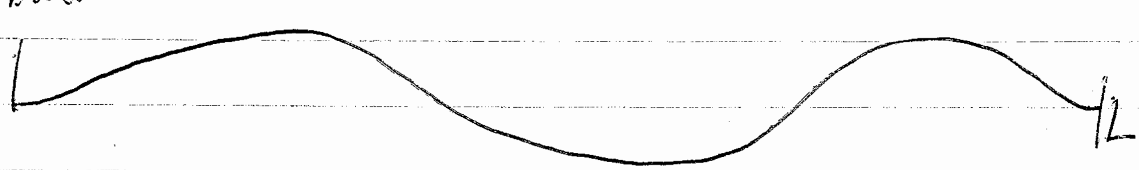
Note that if W is shifted in time, $\langle |F_W(x,\omega)|^2 \rangle$ is unchanged.

Say we have $[0,L]$ filled with data. Examples of W:





between



29.

We calculate something more general: $\langle F_{W_1}(x,\omega)\overline{F_{W_2}(x,\omega)}\rangle$.

$$F_{W_1}(x,\omega)=\int_{-\infty}^{\infty}x(t)w_1(t)e^{-i\omega t}dt=\frac{1}{2\pi}\int x(t_1)\hat{\tilde{W}}_1(v_1)e^{-i\omega t_1}e^{iv_1t_1}dv_1dt_1$$

So $\langle F_{W_1}(x,\omega)\overline{F_{W_2}(x,\omega)}\rangle=$

$$\left\langle\frac{1}{(2\pi)^2}\iiiint_{v_1v_2t_1t_2}x(t_1)x(t_2)\hat{\tilde{W}}_1(v_1)\overline{\hat{\tilde{W}}_2(v_2)}e^{-i\omega t_1+i\omega t_2}e^{iv_1t_1}e^{-iv_2t_2}dv_1dv_2dt_1dt_2\right\rangle$$

Put $t=t_1$, $\tau=t_2-t_1$.

$$=\left\langle\frac{1}{(2\pi)^2}\iiiint_{v_1v_2t_1\tau}x(t)x(t+\tau)\hat{\tilde{W}}_1(v_1)\overline{\hat{\tilde{W}}_2(v_2)}e^{i\omega\tau}e^{iv_1t}e^{-iv_2(t+\tau)}dv_1dv_2dtd\tau\right\rangle$$

$$=\frac{1}{(2\pi)^2}\iiiint_{v_1v_2t\tau}c(\tau)\hat{\tilde{W}}_1(v_1)\overline{\hat{\tilde{W}}_2(v_2)}e^{it(v_1-v_2)}e^{i\tau(\omega-v_2)}dv_1dv_2dtd\tau$$

But $\frac{1}{2\pi}\int e^{it(v_1-v_2)}dt=\delta(v_1-v_2)$

$$=\frac{1}{2\pi}\iint_{v\tau}c(\tau)\hat{\tilde{W}}_1(v)\overline{\hat{\tilde{W}}_2(v)}e^{i\tau(\omega-v)}dvd\tau$$

But $P_X(\omega)=\int e^{-i\omega\tau}c(\tau)d\tau$

$$=\int P_X(\omega-v)\hat{\tilde{W}}_1(v)\overline{\hat{\tilde{W}}_2(v)}dv$$

So,

(a) $$\langle |F_W(x,\omega)|^2 \rangle = \int P_X(\omega-\nu)\,|\tilde{W}(\nu)|^2\, d\nu$$

and

<u>if</u> $P_X(\omega)$ is flat,

(b) $$\langle F_{W_1}(x,\omega)\,\overline{F_{W_2}(x,\omega)} \rangle = K \int \tilde{W}_1(\nu)\,\overline{\tilde{W}_2(\nu)}\, d\nu$$

$$= 2\pi K \int W_1(t)\, W_2(t)\, dt$$

(a) tells us how $F_W(x,\omega)$ is a biased est. of P.

(b) tells us when different taper-ests are independent.

(b') $$\langle F_W(x,\omega)\,\overline{F_{W_2}(x,\omega)} \rangle \approx 0 \text{ if } \int W_1(t)\, W_2(t)\, dt = 0$$

AND $P_X(\omega)$ is constant where $\tilde{W}_1, \tilde{W}_2(\nu)$ are large.

"Standard" window: take

$$w(t) = \begin{cases} \frac{1}{T}, & |t| < \frac{T}{2} \\ 0, & |t| > \frac{T}{2} \end{cases}$$

[note: different normalization c/w earlier]

$$\hat{W}(\omega) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\omega t}\,dt = \left.\frac{e^{-i\omega t}}{-i\omega T}\right|_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{2}{\omega T}\,\frac{e^{-i\omega T/2} - e^{i\omega T/2}}{-2i}$$

$$= \frac{\sin(\omega T/2)}{\omega T/2}$$

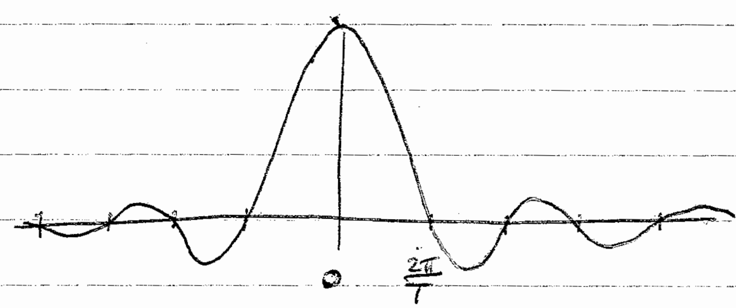
$$= \operatorname{sinc}(\omega T/2).$$

Behavior of sinc: $\operatorname{sinc}(0) = 1$.

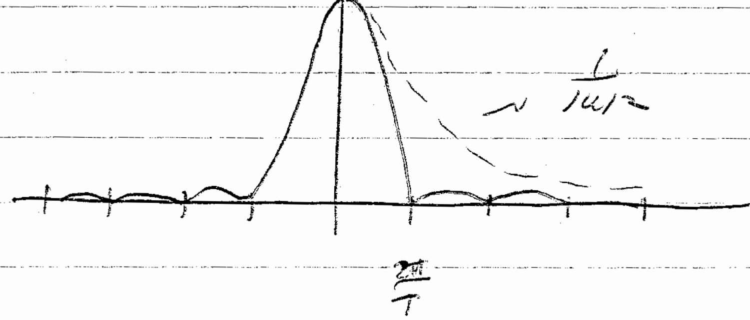
$|\operatorname{sinc}(u)| < \frac{1}{u}$,

$|\operatorname{sinc}(u)| \sim \frac{1}{u}$ at $u = (n+\frac{1}{2})\pi$

$\hat{W}(\omega)$:

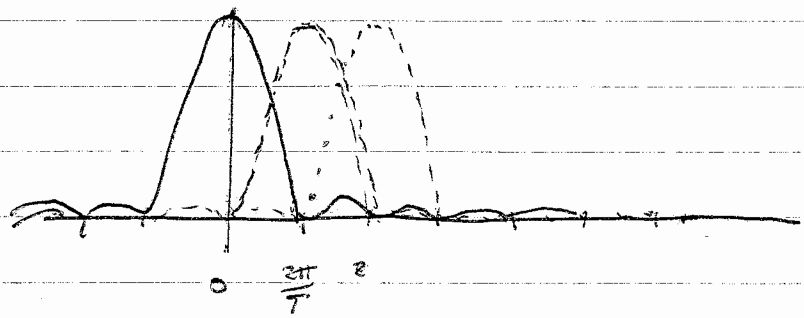


$|\hat{W}(\omega)|^2$:



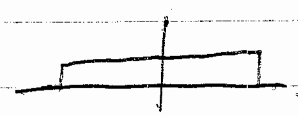
So, we can view the spectral estimate as a bank of "filters", one at each frequency $\frac{2\pi m}{T}$, looking like

$|\hat{W}(\omega)|^2$ but translated to center at $\nu = \frac{2\pi m}{T}$:



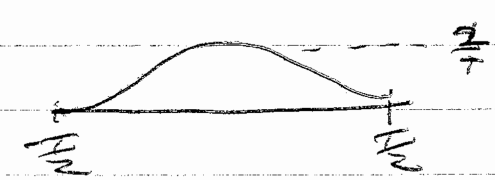
Each filter drops to 0 at other freq's $\nu \pm \frac{2\pi h}{T}$, but is substantial, up to $\frac{1}{|\nu-\nu_0|^2}$, in between.

This would be fine if the power spectrum only had power at $\frac{2\pi h}{T}$ $(h = 0, 1, 2, \dots)$, but it doesn't (or, this is at least a bad assumption).

The "ringing" comes from the sharp edges of $W_R =$  .

So how about, $W_{CB} = \begin{cases} \frac{1}{T}\left(1+\cos\frac{2\pi t}{T}\right) & |t| < \frac{T}{2} \\ 0 & \text{, else} \end{cases}$

$W_{CB} =$



CB = "cosine Bell", "Welch", "Hanning"

33.

Since $W_{CB} = W_R \cdot \left(1 + \cos \frac{2\pi t}{T}\right)$

$$= W_R \left(1 + \frac{1}{2}\left(e^{\frac{2\pi i t}{T}} + e^{-\frac{2\pi i t}{T}}\right)\right)$$

$$\hat{W}_{CB}(\omega) = \hat{W}_R(\omega) * \left[\delta(\omega) + \frac{1}{2}\delta\left(\omega - \frac{2\pi}{T}\right) + \frac{1}{2}\delta\left(\omega + \frac{2\pi}{T}\right)\right]$$

$$= \operatorname{sinc} \frac{\omega T}{2} + \frac{1}{2}\operatorname{sinc}\left(\left(\omega - \frac{2\pi}{T}\right)\frac{T}{2}\right)$$
$$+ \frac{1}{2}\operatorname{sinc}\left(\left(\omega - \frac{2\pi}{T}\right)\frac{T}{2}\right)$$

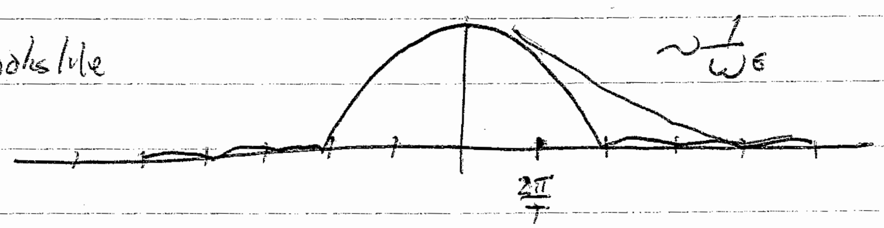
$$= \frac{2}{\omega T} \sin \frac{\omega T}{2} - \frac{1}{2} \sin \frac{\omega T}{2}\left(\frac{1}{\frac{\omega T}{2} + \pi} + \frac{1}{\frac{\omega T}{2} - \pi}\right)$$

$$= \sin \frac{\omega T}{2} \cdot \left[\frac{2}{\omega T} - \frac{2}{\omega T - 4\pi^2/\omega T}\right].$$

For large ω, $|\hat{W}_{CB}(\omega)| \sim \frac{1}{\omega^3}$, so $|\hat{W}_{CB}(\omega)|^2 \sim \frac{1}{\omega^6}$.

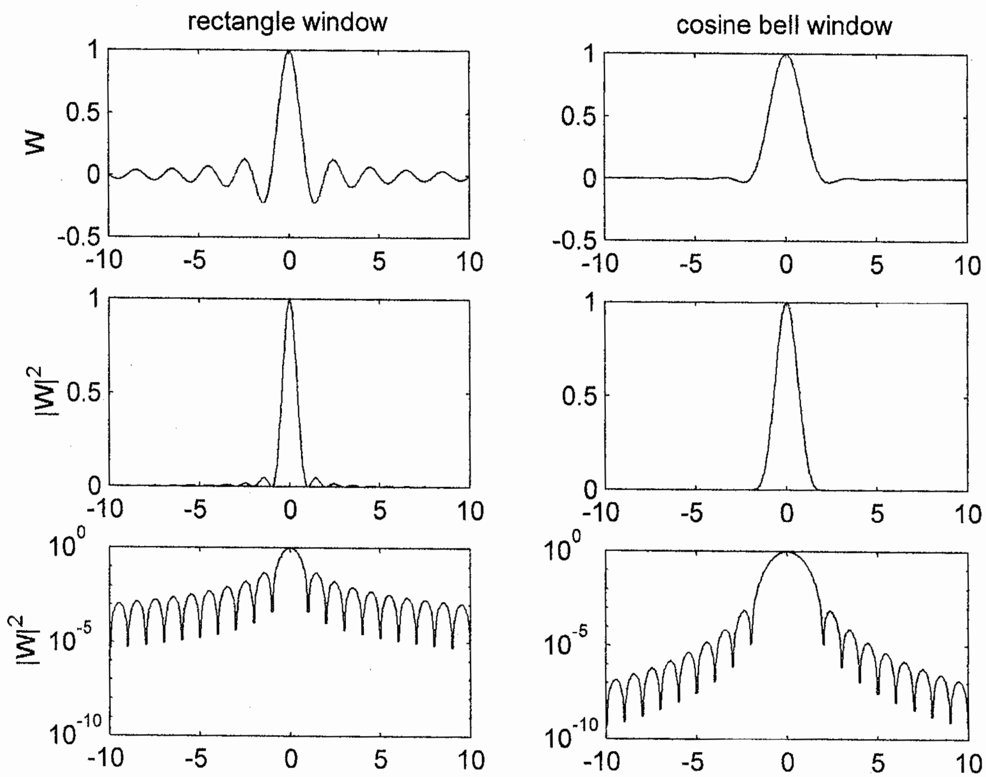
But note $W_{CB}\left(\frac{2\pi}{T}\right) \neq 0$, so

$|W_{CB}|^6$ looks like



We've made the far-freq rolloff faster, but we've made the central band 2x wide.

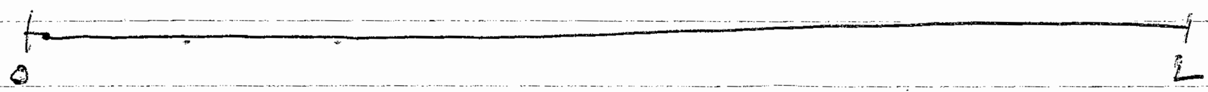
Many other such "windows"!



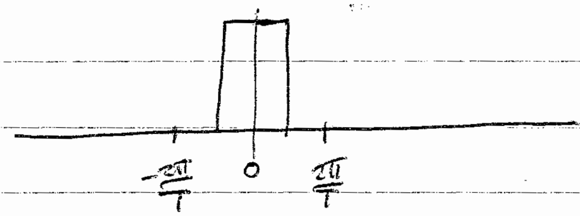
rectangle window
cosine bell window
W
|W|^2
|W|^2
1
0.5
0
-0.5
-10
-5
0
5
10
10^0
10^-5
10^-10

sincdemo

B's insight: Use the whole data length.
(Thompson)

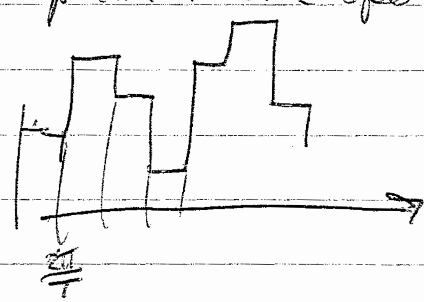


and "windows" needn't be ≥ 0. The "optimal" window (taper) should have a Fourier transform like

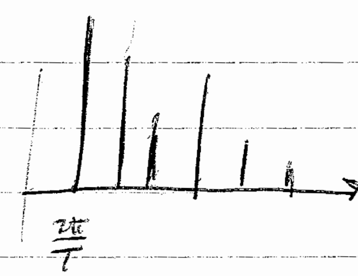


We could choose $T = L$ & get low-quality high-res spectra, but we could also choose $T = L/k$ (k traditionally odd) & get spectra at a lower resolution, but with higher quality. And we've minimized the leakage.

In terms of a Gaussian prior: Assume spectrum looks like

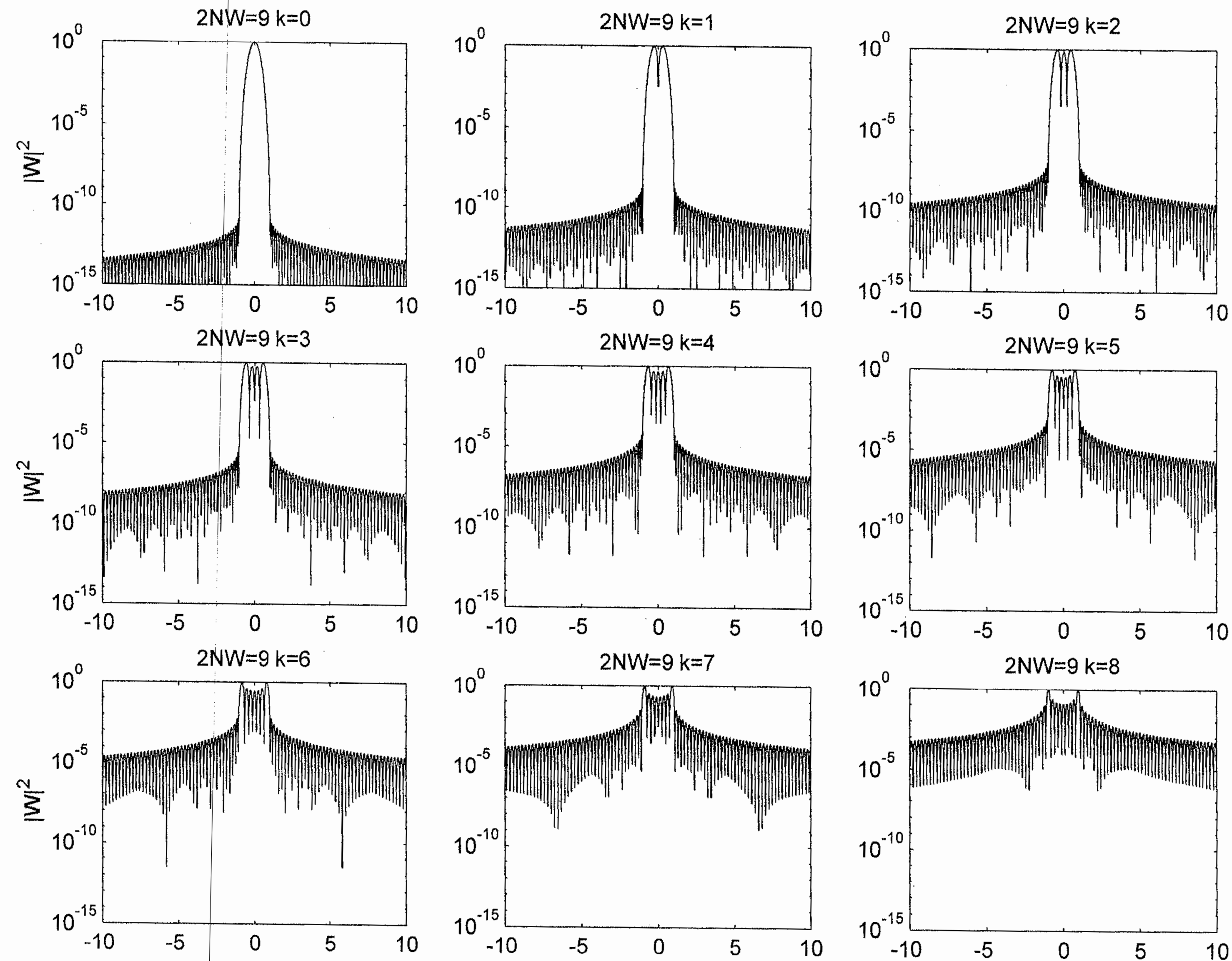


, not



Choose $T=L$: Expect ~ 1 "window" that is confined to L, and with its F.T. confined to $|\omega| \leq \frac{2\pi}{L}$

Choose $T = \frac{L}{k}$: Expect $\sim k$ "windows" that are confined to L, & have their F.T. confined to $|\omega| \leq \frac{2\pi}{T} = \left(\frac{2\pi}{L}\right)k$.


2NW=9 k=0
2NW=9 k=1
2NW=9 k=2
2NW=9 k=3
2NW=9 k=4
2NW=9 k=5
2NW=9 k=6
2NW=9 k=7
2NW=9 k=8
|W|^2
10^0
10^-5
10^-10
10^-15
-10
-5
0
5
10

dpsswdemo

Once we have chosen k, & found the windows (tapers) $\varphi_1, \ldots, \varphi_k$ then we estimate

$$P_X(\omega) \approx \frac{1}{L}\frac{1}{k}\sum_{m=1}^{k}\left|\int_0^L e^{-i\omega t}\varphi_m(t)\,x(t)\,dt\right|^2$$

where we have chosen the normalization $\int_0^L \varphi_m^2(t)dt = L$

to match $\int_0^L 1\,dt = L$

if spectrum is $\sim$ flat, then each estimate

$$\frac{1}{L}\left|\int_0^L e^{-i\omega t}\varphi_m(t)\,x(t)\,dt\right|^2 \text{ is independent.}$$

Immediately leads to cross-spectral estimate, e.g.,

$$C_{xy}(\omega) \approx \frac{1}{L}\cdot\frac{1}{k}\sum_{m=1}^{k}\left(\int_0^L e^{-i\omega t}\varphi_m(t)x(t)dt\right)\overline{\left(\int_0^L e^{-i\omega t}\varphi_m(t)y(t)dt\right)}$$

The φ's, or, more properly, $\frac{\varphi_m}{\sqrt{L}}$ (orthonormal) are

The "prolate spheroidal functions", a classical family of orthogonal functions.

Corrections on φ_m for use on a discrete lattice are the "discrete prolate spheroidal sequences" (dpss.m)

Properties of the φ's:

Consider two operators: $(Df)(x) = \begin{cases} f(x), & |x| \le \frac{L}{2} \\ 0, & |x| > \frac{L}{2} \end{cases}$

+ B, defined in the Fourier domain:

$$(B\tilde{f})(\omega) = \begin{cases} \tilde{f}(\omega), & |\omega| \le \frac{\pi}{T} \\ 0, & |\omega| > \frac{\pi}{T} \end{cases}$$

Argue that our tapers are functions ψ s.t. $DB\psi = \lambda\psi$, for λ near 1.

$DB\psi = \lambda\psi \Rightarrow \psi$ is 0 outside of $|x| \le \frac{L}{2}$.
Hence $\psi = D\psi$,
and
$DBD\psi = \lambda\psi$.

Conversely, an eigenfunction of DBD is an eigenfunction of DB.
But we need to show that $\tilde{\psi}$ is confined to $|\omega| < \frac{\pi}{T}$, or nearly so.

Key is that B + D are related by Fourier transform, with care taken about scale + parity.

Scaled symmetrized F.T.:

$$(F_\gamma g)(y) = \frac{1}{\sqrt{2\pi\gamma}} \int e^{-ixy/\gamma} g(x)\,dx$$

Note $F_\gamma^{-1} = F_\gamma^*$, and $F_\gamma g(y) = \frac{1}{\sqrt{2\pi\gamma}} \hat{g}(\gamma y)$.

Want to choose γ so that $|y| = \frac{L}{2} \Leftrightarrow \gamma|y| = \frac{\pi}{T}$.

$$\gamma \frac{L}{2} = \frac{\pi}{T}, \quad \gamma = \frac{2\pi}{LT}.$$

With this choice, $B = F^{-1} D F$, & $F B F^{-1} = D$

Also, $F^{-1} = FP$, where $Pg(x) = g(-x)$.

$BP = PB$, $DP = PD$,
but $FP \neq PF$.

$$\begin{aligned} BDF = BF(F^{-1}DF) &= BFP \cdot P \cdot F^{-1}DF \\ &= BFP \cdot P \cdot B \\ &= F^{-1}F\, BFP \cdot P \cdot B \\ &= F^{-1}F \cdot BF^{-1}\, P \cdot B \\ &= F^{-1}\, DPB \\ &= FPDPB = FDB \end{aligned}$$

So $BDF\psi = FDB\psi = F\lambda\psi$, i.e.,
$F\psi$ is an eigenfunction of BD.

Other notes:

$D^2 = D$, $B^2 = B$, and D & B are both self-adjoint

$(Df, g) = (f, Dg)$ trivially in the time domain

$(Bf, g) = (f, Bg)$ trivially in the freq domain

BD, DB Not self adjoint but

BDB, DBD are s-a.

$(BDBf, g) = (DBf, Bg) = (Bf, DBg) = (f, BDBg)$ etc.

"Double Orthogonality":

Say ψ_1, ψ_2 eigenfns of DB, $\bar{c}$ eiv's $\lambda_1 \neq \lambda_2$

$(B\psi_1, B\psi_2) = \frac{1}{\lambda_1}(BDB\psi_1, B\psi_2)$

$\hookrightarrow = \frac{1}{\lambda_2}(B\psi_1, BDB\psi_2) = \frac{1}{\lambda_2}(B\psi_1, (BDB)B\psi_2)$

$= \frac{1}{\lambda_2}((BDB)B\psi_1, B\psi_2)$

$= \frac{1}{\lambda_2}(BDB\psi_1, B\psi_2)$

So, if $\lambda_1 \neq \lambda_2$, $(B\psi_1, B\psi_2) = 0$.

$DB\psi_1 = \lambda\psi_1$, so, $B\psi_1 = \frac{1}{\lambda}\psi_1$ on $|x| \leq \frac{L}{2}$.

It is not trivial that these φ's are sol'ns of the same 2nd order ODE that defines the prolate spheroidal f'ns

(Slepian 1940)

Asymptotic rel'n to Hermite functions also not trivial.

Define $D_{gau,L} f(x) = f(x)\, e^{-x^2/2L^2}$

$$B_{gau,T} \tilde{f}(\omega) = \tilde{f}(\omega)\, e^{-\omega^2/2(\pi/T)^2}$$

$$D_{gau,L}^{\;2} = D_{gau,\,L/\sqrt{2}} \qquad (\text{but } D^2 = D)$$

$$B_{gau,T}^{\;2} = B_{gau,\,T/\sqrt{2}} \qquad (\text{but } B^2 = B)$$

The Hermite functions are eigenfunctions of $B_{gau}^{\frac{1}{2}} D_{gau} B_{gau}^{\frac{1}{2}}$

asymptotically, as $LT \gg k^2$ (k = eigen f'n #)

Hermites $\sim$ Slepians

Details in V & K, on web.