Power Spectra IV - Nonstationary Processes

Can we relax the assumption that the ensemble R is "stationary"?

Now $\langle X(t)X(t+\tau)\rangle$ depends both on t & τ.

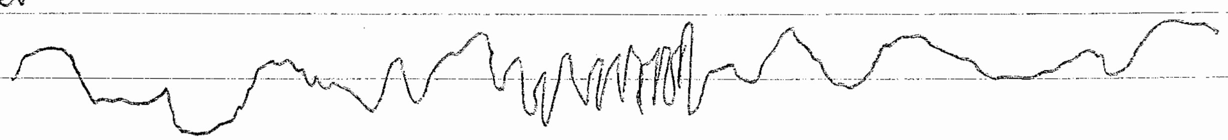
But direct estimation in the time-domain is still problematic, since

$$\langle x(t)x(t+\tau)\rangle$$
$$\langle x(t+\tau)\, x(t+\tau')\rangle$$
and $$\langle x(t) \quad x(t+\tau')\rangle$$

are correlated.

And, in the frequency domain, the problem appears to be undefined!

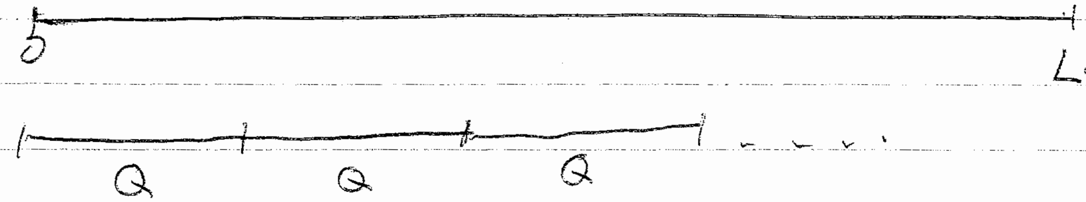
Consider

$s(t) =$ 

We could describe this as an early domain of low freq, then high freq, then low freq. OR, rapidly-varying activity at very low frequencies.

A nonstationary Gaussian process is uniquely described by its 2° moments $\langle x(t)x(t+\tau)\rangle$ - can get these by determining the maxent distribution-
The issue is, how we describe these as a temporal evolution of a slowly-changing Gaussian "stationary" process

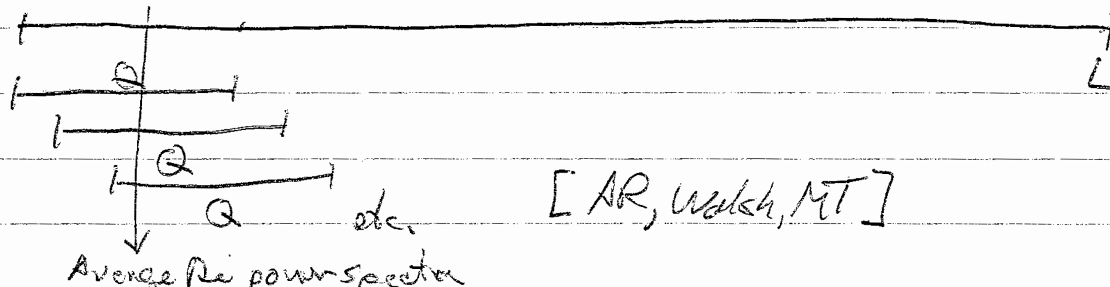
Method I. Choose a timescale of quasi-stationarity, Q, & estimate the P.S. in each segment of length Q



[AR, Welch, MT]

Method II

Sliding Windows



[AR, Welch, MT]

Method II':

[AR] Average the coefficients a_k calculated from each segment

$$y_n = x_n + \sum_{k=1}^{S} a_{k;n_0} y_{n-k}$$

prior to estimating a local spectrum as

$$P_y(\omega) = \frac{|X(\omega)|^2}{|1 - A_{n_0}(\omega)|^2} \quad \left[A_{n_0}(\omega) = \sum a_k e^{j\omega k} \right]$$

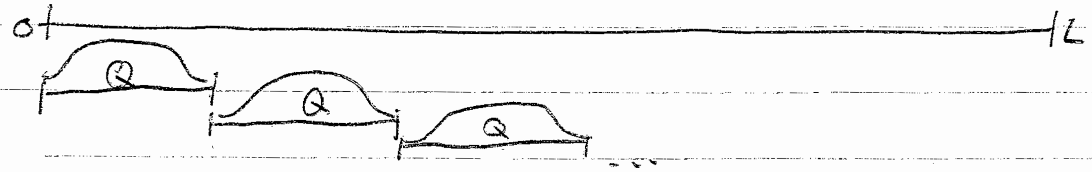
Get a different answer; since P depends nonlinearly on the a's

Method II'': Force the a's to change slowly by adding a penalty like $\sum_{n_0} | a_{k;n_0} - a_{k;n_0-1} |^2$

II – II′ have all the drawbacks of AR methods in the stationary case, ⊕ additional ad-hoc assumptions [but may be justified by a model]

Method I looks like it might be helped by the MT philosophy.

In stationary case



gives spectral estimate based on non-overlapping segments, which can be viewed (tautologously) as spectral estimate of the entire signal, weighted by the (square of) the windowing function.

We previously had the MT estimate (p. 35)

$$P_X(\omega) \approx \frac{1}{L}\,\frac{1}{K}\sum_{m=1}^{K}\left|\int_0^L e^{-i\omega t}\varphi_m(t)\,x(t)\,dt\right|^2 \text{ with } \int_0^L \varphi_m^2(t)\,dt = L$$

but also, $\int_0^L \varphi_m(t)\varphi_n(t)\,dt = 0$, $m \neq n$ ["double orthogonality"]

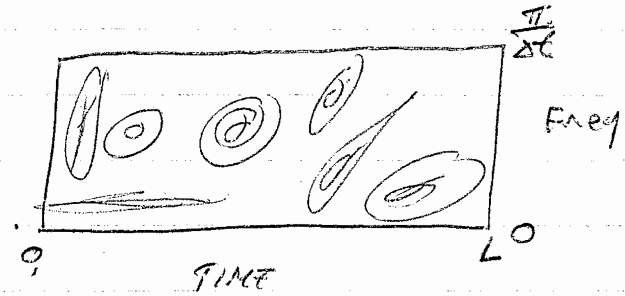
So, for a time-varying estimate, can take

$$P_X(\omega,\tau) = \frac{1}{L}\sum_{m=1}^{K}\left|\int_0^L e^{-i\omega t}\varphi_m(t)\,x(t)\,dt\right|^2 \cdot \frac{\varphi_m(\tau)}{L}$$

93.

What is the meaning of the chosen "K"?

Ideal time-frequency diagram "spectrogram"



MAX resolution on frequency axis $= \frac{2\pi}{L}$. $\left(\frac{\pi}{\Delta t} / \left(\frac{2\pi}{L}\right) = \frac{L}{2\Delta t}\right.$ points

" " " time axis $= \Delta t$. $\left(\frac{L}{\Delta t}\right.$ points)

Total # of points =

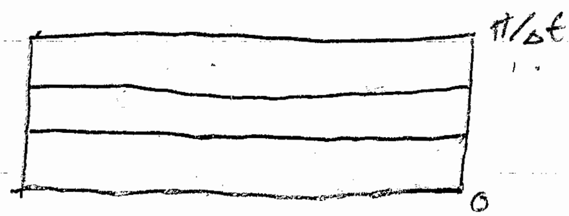
$$\underset{\text{TIME}}{\frac{L}{\Delta t}} \cdot \underset{\text{Freq}}{\frac{L}{2\Delta t}} = \frac{1}{2}\left(\frac{L}{\Delta t}\right)^2$$

But we only have $\frac{L}{2\Delta t}$ data points. Cannot achieve the "ideal" spectrogram.

Spectrum ~

in steps of $\frac{2\pi}{L}$ $(K=1)$

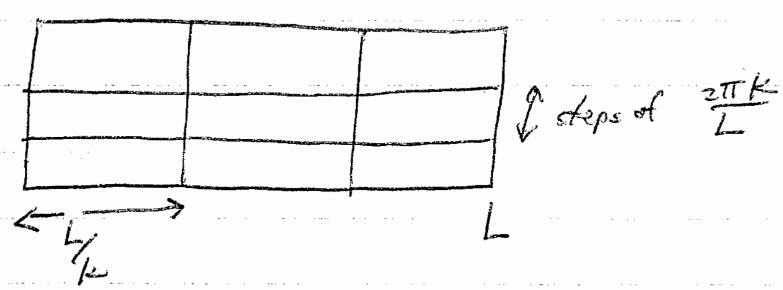
K estimates per pixel



in steps of $\frac{2\pi K}{L}$

Now, we want to use the K-resolution to look at time-variation, sacrificing the robustness of the estimate:

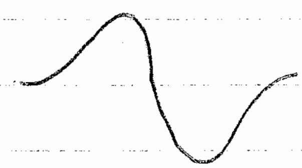
1 estimate per pixel



Can improve estimate by having more examples of $x(t)$ and/or averaging neighboring pixels

The parcellation of the spectrogram need not be restricted to these schemes.

Wavelet analysis: Choose a "wavelet" $g(t)$, and calculate $\left|\int x(t)\, g(\alpha_0(t-t_0))\, dt\right|^2$ for multiple scales α_0, and multiple start-times t_0.

Typically, $g(t)$ taken to look like 

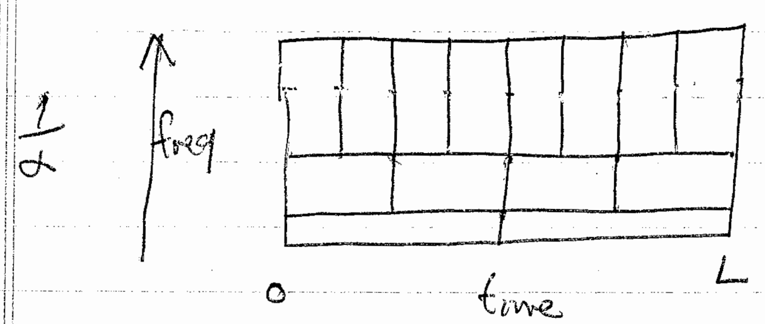
or, 

(Gabor = (sine or cos) · Gaussian;
Hermite fn = specific polynomial · Gaussian)

For t_0, t_1 close or α_0, α_1 close, $g(\alpha_0(t-t_0))$ and $g(\alpha_1(t-t_1))$ will provide highly correlated estimates - but for α_0, α_1 and/or t_0, t_1 widely separated, they are independent

Increasing α yields higher frequencies & shorter times for the duration of the envelope.

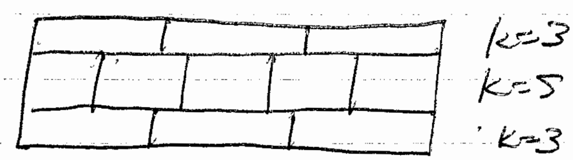
To the extent that sufficiently different α_0's and/or f_0's yield independent estimates, the wavelet approach corresponds to parcellation like this:



Tiles have equal area: doubling the value of α_0 gives 2x the freq resolution, ½x the temporal resolution.

The number of ripples in the "mother" wavelet controls the shape of the $\alpha_0 = 1$ -tile

One can achieve this with the MT method by choosing different values of "K" for different frequency bands (or, even greater flexibility:



etc.

In the MT method, estimates in each pixel are very nearly independent (the slight leakage ~ "very nearly")
on p. 34A