Algebraic Overview

Homework #1 (2008) Answers

Q1: Eigenvectors of some linear operators in matrix form. In each case, find the eigenvalues, the eigenvectors, the dimensions of the eigenspaces, and whether a basis can be chosen from the eigenvectors.

A. $A = \begin{pmatrix} q & 1 \\ 0 & q \end{pmatrix}$.

First, use the determinant to find the eigenvalues.

$\det(A - zI) = \det\begin{pmatrix} q-z & 1 \\ 0 & q-z \end{pmatrix} = (q-z)^2$, so the only eigenvalue of A is q.

Say V has basis elements e_1 and e_2, expressed as columns $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $Ae_1 = qe_1$, and $Ae_2 = e_1 + qe_2$. $Ae_1 = qe_1$ means that e_1 (and any multiple of it) is an eigenvector with eigenvalue q.

To look for other eigenvectors v with eigenvalue q Say $v = ae_1 + be_2$. Then, $Av = qv$ implies $aqe_1 + bqe_2 = qv = Av = aqe_1 + b(e_1 + qe_2) = (aq+b)e_1 + bqe_2$. Since e_1 and e_2 are linearly independent (they form a basis), their coefficients must be equal. For e_2, this is guaranteed $bq = bq$, but for e_1, this implies that $aqe_1 = (aq+b)e_1$, which in turn means $b = 0$. That is, $v = ae_1 + be_2$ must be a multiple of e_1, i.e., there are no other eigenvectors.

So there is one eigenvalue q , whose eigenspace has dimension 1, spanned by the eigenvector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since A operates in a two-dimensional vector space, the eigenvectors cannot form a basis.

B. $B = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ *(assume* $a > b > c > 0$ *). Do the eigenvectors form a basis? Hint: Observe that B commutes with* $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$*, and find the eigenvalues and eigenvectors of T.*

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Carrying out the hint: Observe that B and T commute. $BT = \begin{pmatrix} c & a & b \\ b & c & a \\ a & b & c \end{pmatrix} = TB$.

Eigenvalues and eigenvectors for T:

$\det(T - zI) = \det\begin{pmatrix} -z & 1 & 0 \\ 0 & -z & 1 \\ 1 & 0 & -z \end{pmatrix} = 1 - z^3$. This has solutions $z = \{1, \omega, \omega^2\}$ where $\omega = e^{2\pi i/3}$, a complex cube root of 1. (Think of where the solutions $z = \{1, \omega, \omega^2\}$ lie on the complex plane.)

There is a more insightful approach to finding the eigenvalues that avoids calculating the determinant: Note that multiplying B by T permutes the roles of a, b, and c. Note also that T is a special case of a B, with b=1, a=0, c=0. So it follows that $T^3 = I$. From this, it follows that for any eigenvalue λ of T, that $\lambda^3 = 1$.

Now, find the eigenvectors for T: Say $v = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = t_1 e_1 + t_2 e_2 + t_3 e_3$, for bases e_1, e_2, e_3 defined as in Q1A. Then $Tv = \begin{pmatrix} t_2 \\ t_3 \\ t_1 \end{pmatrix} = t_2 e_1 + t_3 e_2 + t_1 e_3$. So, $Tv = \lambda v$ implies $t_2 = \lambda t_1$, $t_3 = \lambda t_2$, and $t_1 = \lambda t_3$. That is, $v = t_1 \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}$ and (as we already knew) $\lambda^3 = 1$. So the eigenvectors corresponding to the three eigenvalues $z = \{1, \omega, \omega^2\}$ are $v_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$. (I've numbered them starting at 0 so that one can conveniently write $v_m = \begin{pmatrix} 1 \\ \omega^m \\ \omega^{2m} \end{pmatrix}$.)

So there are three distinct eigenvalues for T and three eigenspaces for T. Each have dimension 1, and the strong result about eigenvectors of commuting operators applies.

All that is left to do is to find the eigenvalues for A associated with the three v_m's.

$$Bv_m = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \begin{pmatrix} 1 \\ \omega^m \\ \omega^{2m} \end{pmatrix} = \begin{pmatrix} a + b\omega^m + c\omega^{2m} \\ c + a\omega^m + b\omega^{2m} \\ b + c\omega^m + a\omega^{2m} \end{pmatrix}$$, so the eigenvalue associated with v_m is $a + b\omega^m + c\omega^{2m}$.

C. $C = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

$\det(C - zI) = \det\begin{pmatrix} \cos\theta - z & -\sin\theta \\ \sin\theta & \cos\theta - z \end{pmatrix} = z^2 - 2z\cos\theta + 1$. Using the quadratic formula, this has roots $z = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$.

So there are two eigenvalues. For the first eigenvalue ($e^{i\theta}$), the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfies $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} e^{i\theta}$. That is, $x\cos\theta + y\sin\theta = xe^{i\theta} = x\cos\theta + ix\sin\theta$, which solves for $\begin{pmatrix} x \\ y \end{pmatrix} \propto \begin{pmatrix} 1 \\ i \end{pmatrix}$. Similarly, for the eigenvalue $e^{-i\theta}$, we find an eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} \propto \begin{pmatrix} 1 \\ -i \end{pmatrix}$. The two eigenspaces are each of dimension 1, and the two eigenvectors form a basis.

D. $D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

In the subspace spanned by e_1 and e_2, D acts like $3I$, so all linear combinations of e_1 and e_2 are eigenvectors of eigenvalue 3.

$De_3 = 2e_3$ so e_3 is an eigenvector of eigenvalue 2.

Within the subspace spanned by e_4 and e_5, D acts like the matrix A of Q1A, with $q = 0$. So e_4 is an eigenvector of eigenvalue 0, and there are no other eigenvectors in the subspace spanned by e_4 and e_5.

So there is a 2-dimensional eigenspace for eigenvalue 3, a 1-dimensional eigenspace for eigenvalue 2, and a 1-dimensional eigenspace for eigenvalue 0. There are only 4 linearly independent eigenvectors, so they cannot form a basis.

Q2: Adjoints, etc.

A. Work in the vector space of finite dimension N over the complex numbers. Use the standard inner product $\langle x,y\rangle=\sum_{k=1}^{N}x_k\overline{y_k}$ *Given an operator A in matrix form (specified by an array* a_{kl}*, so that if* $x=\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix}$, $z=\begin{pmatrix}z_1\\ \vdots\\ z_N\end{pmatrix}$ *and* $z=Ax$*, then* $z_k=\sum_{l=1}^{N}A_{kl}x_l$ *), find the matrix form of its adjoint* A^*.

$$\langle Ax,y\rangle=\sum_{k=1}^{N}\left(\sum_{l=1}^{N}A_{kl}x_l\right)\overline{y_k}=\sum_{k=1,l=1}^{N}x_lA_{kl}\overline{y_k}=\sum_{k=1,l=1}^{N}x_l\overline{\overline{A_{kl}}y_k}=\sum_{l=1}^{N}x_l\overline{w_l}\text{, where } w_l=\sum_{k=1}^{N}\overline{A_{kl}}y_l\ .$$

So, $\langle Ax,y\rangle=\langle x,A^*y\rangle$ for $\left(A^*y\right)_k=\sum_{l=1}^{N}\overline{A_{lk}}y_k$ (we just swapped the roles of *l* and *k*). That is, the matrix form of A^* are the elements $\left(a^*\right)_{kl}=\overline{a_{lk}}$. In words, the adjoint is the conjugate of the transpose.

B. Work in the vector space of complex-valued functions of time, and using the inner product $\langle f,g\rangle=\int_{-\infty}^{\infty}f(t)\overline{g(t)}dt$. *Find the adjoint of the time-translation operator* $\left(D_Tf\right)(t)=f(t+T)$.

$\langle D_Tf,g\rangle=\int_{-\infty}^{\infty}f(t+T)\overline{g(t)}dt=\int_{-\infty}^{\infty}f(t')\overline{g(t'-T)}dt'=\langle f,D_{-T}g\rangle$, where the middle equality follows from a change of variables $t'=t+T$. So $D_T{}^*=D_{-T}$.

C. Set up as in B. Find the adjoint of the linear operator A, where Af is defined by $(Af)(t)=\int_{-\infty}^{\infty}A(t,\tau)f(\tau)d\tau$.

The calculation is precisely analogous to Q2A.

$$\langle Af,g\rangle=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}A(t,\tau)f(\tau)d\tau\right)\overline{g(t)}dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(t,\tau)f(\tau)\overline{g(t)}d\tau dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)\overline{\overline{A(t,\tau)}g(t)}d\tau dt$$

$=\int_{-\infty}^{\infty}f(\tau)\left(\int_{-\infty}^{\infty}\overline{\overline{A(t,\tau)}g(t)dt}\right)d\tau=\int_{-\infty}^{\infty}f(\tau)h(\tau)d\tau$. So $\langle Af,g\rangle=\langle f,A^*g\rangle$ where

$(A^{*}g)(t) = h(t) = \int_{-\infty}^{\infty} \overline{A(\tau,t)}g(\tau)d\tau$. (Note that the roles of t and τ were just swapped.)

That is, if A is specified by $A(t,\tau)$, then $A^{*}(t,\tau) = \overline{A(\tau,t)}$ -- also a conjugate transpose, as in Q2A.