Algebraic Overview

Homework #1 (2008) Answers

Q1: Eigenvectors of some linear operators in matrix form. In each case, find the eigenvalues, the eigenvectors, the dimensions of the eigenspaces, and whether a basis can be chosen from the eigenvectors.

A. $A = \begin{pmatrix} q & 1 \\ 0 & q \end{pmatrix}$.

First, use the determinant to find the eigenvalues.

$\det(A - zI) = \det\begin{pmatrix} q-z & 1 \\ 0 & q-z \end{pmatrix} = (q-z)^2$, so the only eigenvalue of *A* is *q*.

Say *V* has basis elements e_1 and e_2, expressed as columns $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $Ae_1 = qe_1$, and $Ae_2 = e_1 + qe_2$. $Ae_1 = qe_1$ means that e_1 (and any multiple of it) is an eigenvector with eigenvalue *q*.

To look for other eigenvectors *v* with eigenvalue *q* Say $v = ae_1 + be_2$. Then, $Av = qv$ implies $aqe_1 + bqe_2 = qv = Av = aqe_1 + b(e_1 + qe_2) = (aq + b)e_1 + bqe_2$. Since e_1 and e_2 are linearly independent (they form a basis), their coefficients must be equal. For e_2, this is guaranteed $bq = bq$, but for e_1, this implies that $aqe_1 = (aq + b)e_1$, which in turn means $b = 0$. That is, $v = ae_1 + be_2$ must be a multiple of e_1, i.e., there are no other eigenvectors.

So there is one eigenvalue *q* , whose eigenspace has dimension 1, spanned by the eigenvector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since *A* operates in a two-dimensional vector space, the eigenvectors cannot form a basis.

B. $B = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ *(assume* $a > b > c > 0$ *). Do the eigenvectors form a basis? Hint: Observe that B commutes with* $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$*, and find the eigenvalues and eigenvectors of T.*

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Carrying out the hint: Observe that *B* and *T* commute. $BT = \begin{pmatrix} c & a & b \\ b & c & a \\ a & b & c \end{pmatrix} = TB$.

Eigenvalues and eigenvectors for *T*:

$\det(T - zI) = \det\begin{pmatrix} -z & 1 & 0 \\ 0 & -z & 1 \\ 1 & 0 & -z \end{pmatrix} = 1 - z^3$. This has solutions $z = \{1, \omega, \omega^2\}$ where $\omega = e^{2\pi i/3}$, a complex cube root of 1. (Think of where the solutions $z = \{1, \omega, \omega^2\}$ lie on the complex plane.)

There is a more insightful approach to finding the eigenvalues that avoids calculating the determinant: Note that multiplying *B* by *T* permutes the roles of *a*, *b*, and *c*. Note also that *T* is a special case of a *B*, with *b*=1, *a*=0, *c*=0. So it follows that $T^3 = I$. From this, it follows that for any eigenvalue λ of *T*, that $\lambda^3 = 1$.

Now, find the igenvectors for *T*: Say $v = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = t_1 e_1 + t_2 e_2 + t_3 e_3$, for bases e_1, e_2, e_3 defined as in Q1A. Then $Tv = \begin{pmatrix} t_2 \\ t_3 \\ t_1 \end{pmatrix} = t_2 e_1 + t_3 e_2 + t_1 e_3$. So, $Tv = \lambda v$ implies $t_2 = \lambda t_1$, $t_3 = \lambda t_2$, and $t_1 = \lambda t_3$. That is, $v = t_1 \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}$ and (as we already knew) $\lambda^3 = 1$. So the eigenvectors corresponding to the three eigenvalues $z = \{1, \omega, \omega^2\}$ are

$v_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$. (I've numbered them starting at 0 so that one can conveniently write $v_m = \begin{pmatrix} 1 \\ \omega^m \\ \omega^{2m} \end{pmatrix}$.)

So there are three distinct eigenvalues for *T* and three eigenspaces for *T*. Each have dimension 1, and the strong result about eigenvectors of commuting operators applies.

All that is left to do is to find the eigenvalues for *A* associated with the three v_m's.

C. $C = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

$\det(C - zI) = \det\begin{pmatrix} \cos\theta - z & -\sin\theta \\ \sin\theta & \cos\theta - z \end{pmatrix} = z^2 - 2z\cos\theta + 1$. Using the quadratic formula, this has roots $z = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$.

So there are two eigenvalues. For the first eigenvalue ($e^{i\theta}$), the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfies $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} e^{i\theta}$. That is, $x\cos\theta + y\sin\theta = xe^{i\theta} = x\cos\theta + ix\sin\theta$, which solves for $\begin{pmatrix} x \\ y \end{pmatrix} \propto \begin{pmatrix} 1 \\ i \end{pmatrix}$. Similarly, for the eigenvalue $e^{-i\theta}$, we find an eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} \propto \begin{pmatrix} 1 \\ -i \end{pmatrix}$. The two eigenspaces are each of dimension 1, and the two eigenvectors form a basis.

D. $D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

In the subspace spanned by e_1 and e_2, D acts like $3I$, so all linear combinations of e_1 and e_2 are eigenvectors of eigenvalue 3.

$De_3 = 2e_3$ so e_3 is an eigenvector of eigenvalue 2.

Within the subspace spanned by e_4 and e_5, D acts like the matrix A of Q1A, with $q = 0$. So e_4 is an eigenvector of eigenvalue 0, and there are no other eigenvectors in the subspace spanned by e_4 and e_5.

So there is a 2-dimensional eigenspace for eigenvalue 3, a 1-dimensional eigenspace for eigenvalue 2, and a 1-dimensional eigenspace for eigenvalue 0. There are only 4 linearly independent eigenvectors, so they cannot form a basis.

Q2: Adjoints, etc.

A. Work in the vector space of finite dimension N over the complex numbers. Use the standard inner product $\langle x,y\rangle = \sum_{k=1}^{N} x_k \overline{y_k}$ Given an operator A in matrix form (specified by an array a_{kl}, so that if $x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}$ and $z = Ax$, then $z_k = \sum_{l=1}^{N} A_{kl} x_l$), find the matrix form of its adjoint A^*.

$\langle Ax,y\rangle = \sum_{k=1}^{N}\left(\sum_{l=1}^{N} A_{kl} x_l\right)\overline{y_k} = \sum_{k=1,l=1}^{N} x_l A_{kl}\overline{y_k} = \sum_{k=1,l=1}^{N} x_l \overline{\overline{A_{kl}} y_k} = \sum_{l=1}^{N} x_l \overline{w_l}$, where $w_l = \sum_{k=1}^{N} \overline{A_{kl}} y_l$.

So, $\langle Ax,y\rangle = \langle x, A^* y\rangle$ for $\left(A^* y\right)_k = \sum_{l=1}^{N} \overline{A_{lk}} y_k$ (we just swapped the roles of l and k). That is, the matrix form of A^* are the elements $\left(a^*\right)_{kl} = \overline{a_{lk}}$. In words, the adjoint is the conjugate of the transpose.

B. Work in the vector space of complex-valued functions of time, and using the inner product $\langle f,g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt$. Find the adjoint of the time-translation operator $\left(D_T f\right)(t) = f(t+T)$.

$\langle D_T f, g\rangle = \int_{-\infty}^{\infty} f(t+T)\overline{g(t)}dt = \int_{-\infty}^{\infty} f(t')\overline{g(t'-T)}dt' = \langle f, D_{-T} g\rangle$, where the middle equality follows from a change of variables $t' = t+T$. So $D_T{}^* = D_{-T}$.

C. Set up as in B. Find the adjoint of the linear operator A, where Af is defined by $(Af)(t) = \int_{-\infty}^{\infty} A(t,\tau) f(\tau) d\tau$.

The calculation is precisely analogous to Q2A.

$\langle Af, g\rangle = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} A(t,\tau) f(\tau) d\tau\right)\overline{g(t)}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(t,\tau) f(\tau)\overline{g(t)}d\tau dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)\overline{\overline{A(t,\tau)}g(t)}d\tau dt$

$= \int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} \overline{\overline{A(t,\tau)}g(t)dt}\right)d\tau = \int_{-\infty}^{\infty} f(\tau)h(\tau)d\tau$. So $\langle Af, g\rangle = \langle f, A^* g\rangle$ where

$(A^* g)(t) = h(t) = \int_{-\infty}^{\infty} \overline{A(\tau,t)} g(\tau) d\tau$. (Note that the roles of t and τ were just swapped.)

That is, if A is specified by $A(t,\tau)$, then $A^*(t,\tau) = \overline{A(\tau,t)}$ -- also a conjugate transpose, as in Q2A.