① Homework Week 1 (2008–2009)
[groups, Fields, VS pgs 1–4]

Q1 Display two groups with 4 elements.

First group is $\mathbb{Z}_4$, rotation group of square (or, addition mod 4)

G_1:

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Second group: G_2

G_2:

	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

They are abstractly different: G_2 has 3 elements of order 2 $(1+1 = 2+2 = 3+3 = 0)$
while G_1 has only 1 element of order 2, and two of order 4.

A more useful presentation of G_2:

The 4 elements: $(0,0) =$ "0"
$(0,1) =$ "1"
$(1,0) =$ "2"
$(1,1) =$ "3"

Add them, element by element, in $\mathbb{Z}_2$.

For example
"1" + "3" $= (0,1)+(1,1) = (0+1, 1+1)$
$= (1,0) =$ "2"

So $G_2 =$ two parallel copies of $\mathbb{Z}_2$, i.e., $G_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

② HOMEWORK, WEEK 1 (2008-9)
[groups, Fields, VS pgs 1-4]

Q1 Which are groups? Commutative or not? Finite? Discrete?

A. 2×2 integer matrices with determinant 1, under multiplication
(G1) Associativity follows from associativity for matrix multiplication
(G2) Identity. $\begin{pmatrix}1&0\\0&1\end{pmatrix}$
(G3) Inverse? Inverse of $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ is $\frac{1}{\det\begin{pmatrix}a&b\\c&d\end{pmatrix}}\cdot\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$

$\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=1$ by hypothesis.

Not commutative. $\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a+c&b+d\\c&d\end{pmatrix}$
but $\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}a&a+b\\c&c+d\end{pmatrix}$
Infinite. $\begin{pmatrix}1&n\\0&1\end{pmatrix}$ is in G.
Discrete.

B. Functions $f(t)$ under $+$:
(G1) $[(f+g)+h](t)=(f+g)(t)+h(t)=f(t)+g(t)+h(t)$
(G2) $z(t)=0$ is the identity.
(G3) $-f(t)$ is the inverse of $f(t)$.
Commutative since $(f+g)(t)=f(t)+g(t)=g(t)+f(t)$.
Infinite. $f(t)=$ anything is in G.
Continuous: "small" $f(t)$'s are in G.

C. Functions $f(t)$ under convolution
$(f*g)(t)=\int f(t-z)\,g(z)\,dz$

③ HW Week 1

(G1) Associative?

$$[(f*g)*h](t) = \int (f*g)(t-z)\,h(z)\,dz$$

$$= \int\left(\int f(t-z-y)\,g(y)\,dy\right)h(z)\,dz$$

$$= \iint f(t-z-y)\,g(y)\,h(z)\,dy\,dz. \qquad (A)$$

$$[f*(g*h)](t) = \int f(t-u)\,(g*h)(u)\,du$$

$$= \int f(t-u)\left(\int g(u-v)\,h(v)\,dv\right)du$$

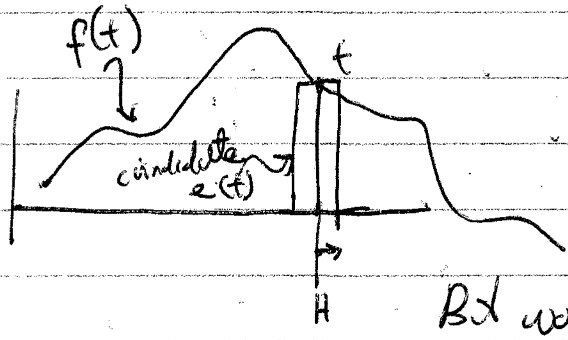
$$= \int f(t-u)\,g(u-v)\,h(v)\,dv\,du \qquad (B)$$

Substitute $u = z+y$, $v = z$ to see $(A) = (B)$.

(G2) Identity? No - would need

$$f(t) = \int f(t-z)\,e(z)\,dz$$

but this expression averages f as weighted by $e(z)$



If $e_H(z) = \begin{cases} \frac{1}{2H}, & |z| < H \\ 0, & \text{otherwise} \end{cases}$

this approximately works.

But would need $H \to 0$.

HW Week 1.

NB The formal limit of e_H, for $H \to 0$ is the "delta-function" $\delta(h)$.

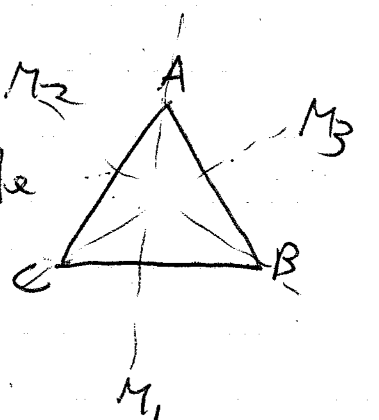
Think of $\delta(h)$ as an infinitely narrow impulse of unit area.

$$f(t) = \int f(t-z)\,\delta(z)\,dz \quad \text{by definition}$$

but δ is not a real-valued function.

Bonus Problem.

Find the smallest non-commutative group.
We stated (easy proof!) that if a group has a prime size it must be $\mathbb{Z}_p$. (Hint: say it had no elements of order p.)
We also found two groups of order 4, and it is easy to show that there are no more. (Consider whether the group has an element of order 4 → it must be G_1, and if it does not → it must be G_2).
So next is size 6.



Rotations AND mirror-flips of the triangle

e	R_1	R_2	M_1	M_2	M_3
$A \to A$	$A \to B$	$A \to C$	$A \to A$	$A \to C$	$A \to B$
$B \to B$	$B \to C$	$B \to A$	$B \to C$	$B \to B$	$B \to A$
$C \to C$	$C \to A$	$C \to B$	$C \to B$	$C \to A$	$C \to C$

$R_1 \circ M_1 = M_2$ but $M_1 \circ R_1 = M_3$. $\circ$ = "followed by"

(5) HW Week 1

Super bonus.
Groups of size 8.

First commutative groups.
Is there an el. of order 8? If so, it must be $\mathbb{Z}_8$. (#1)
No element of order 8, but one of order 4 $\to$ $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ (#2)
No element of order 8 or of order 4 $\to$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. (#3)

Non-commutative.
There Must be an elem of order >2. If not:

$$(ab)^2 = 1 \Rightarrow (ab)^{-1} = ab.$$
But $(ab)^{-1} = b^{-1}a^{-1} = ba$. So a and b must commute.

Start with e, x, x^2, x^3 and consider the possibilities for the others. Choose $y \notin \{e, x, x^2, x^3\}$.
Then the group must be
$$e,\ x,\ x^2,\ x^3,\ y,\ xy,\ x^2y,\ x^3y.$$
All we need to do is determine what is yx. Can't be x^k, otherwise $yx = x^k \Rightarrow y = x^{k-1}$!
So yx must be xy, x^2y, or x^3y.
$yx = xy \Rightarrow$ G would be commutative, $(x^a y^b x^c y^d = x^{a+b} y^{b+d})$
$yx = x^2y \Rightarrow (yxy^{-1})^2 = (x^2)^2 = x^4 = e$ but also
$(yxy^{-1})^2 = (yxy^{-1})(yxy^{-1}) = yx^2y^{-1}$,
and if $yx^2y^{-1} = e$, then $x^2 = e$,
So it must be that $yx = x^3y$.

⑥

What is y^2? either $y^2 = e$, or $(y^2)^2 = y^4 = e$.

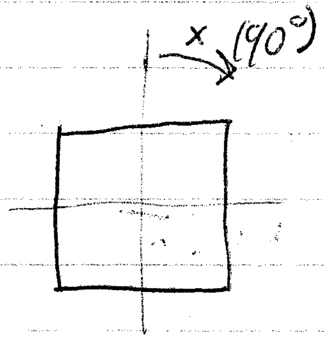
Case 1 (Group 4) $y^2 = e$.

Group is $e \quad x \quad x^2 \quad x^3 \quad y \quad xy, \quad x^2y, \quad x^3y$
and $yx = x^3y$, so group table is

G_4	e	x	x^2	x^3	y	xy	x^2y	x^3y
e	e	x	x^2	x^3	y	xy	x^2y	x^3y
x	x	x^2	x^3	e	xy	x^2y	x^3y	y
x^2	x^2	x^3	e	x	x^2y	x^3y	y	xy
x^3	x^3	e	x	x^2	x^3y	y	xy	x^2y
y	y	x^3y	x^2y	xy	e	x^3	x^2	x
xy	xy	y	x^3y	x^2y	x	e	x^3	x^2
x^2y	x^2y	xy	y	x^3y	x^2	x	e	x^3
x^3y	x^3y	x^2y	xy	y	x^3	x^2	x	e

all of order 2

$x = 90°$ rot
y = mirror flip



Case 2 (Group 5) $y^4 = e$, $\boxed{y^2 = x^2}$
(y^2 cannot be in $\{y, xy, x^2y, x^3y\}$, so y^2 must $= x^2$)

	G_5	e	x	x^2	x^3	y	xy	x^2y	x^3y
e	e	e	x	x^2	x^3	y	xy	x^2y	x^3y
i	x	x	x^2	x^3	e	xy	x^2y	x^3y	y
-1	x^2	x^2	x^3	e	x	x^2y	x^3y	y	xy
$-i$	x^3	x^3	e	x	x^2	x^3y	y	xy	x^2y
j	y	y	x^3y	x^2y	xy	x^2	x	e	x^3
k	xy	xy	y	x^3y	x^2y	x^3	x^2	x	e
$-j$	x^2y	x^2y	xy	y	x^3y	e	x^3	x^2	x
$-k$	x^3y	x^3y	x^2y	xy	y	x	e	x^3	x^2

Quaternions