Groups, Fields, and Vector Spaces

Homework #3 (2008) for pages 9-16 of notes-answers

Q1: Coordinate-dependent isomorphisms of vector spaces.

Given:
Vector space $V$ (with elements $v, ...$) and a basis set $\{e_1, e_2, ..., e_M\}
Vector space $W$ (with elements $w, ...$) and a basis set $\{f_1, f_2, ..., f_N\}

We’ll construct two vector spaces of dimension $M \times N$, $V \otimes W$ and $\text{Hom}(V, W)$. We will then see what happens to the coordinates in these vector spaces when we change basis sets in $V$ and $W$ to new basis sets, $\{e'_1, e'_2, ..., e'_M\}$ for $V$ and $\{f'_1, f'_2, ..., f'_N\}$ for $W$. The new and old basis sets are related by $e_i = \sum_{k=1}^{M} A_{ik} e'_k$ and $f_j = \sum_{i=1}^{N} B_{ij} f'_i$.

A. As discussed in class (notes pg 16), the vector space $V \otimes W$ has a basis set $\{e_i \otimes f_j, ..., e_M \otimes f_N\}$, i.e., any element $z$ of $V \otimes W$ can be written in coordinates as $z = \sum_{i=1,j=1}^{M,N} z_{ij} (e_i \otimes f_j)$, for some $M \times N$ array of scalars $z_{ij}$.

The exercise is to express $z = \sum_{i=1,j=1}^{M,N} z_{ij} (e_i \otimes f_j)$ in terms of the new basis set for $V \otimes W$, namely as a sum $z = \sum_{k=1,j=1}^{M,N} z'_{kj} (e'_k \otimes f'_j)$. That is, find $z'_{kj}$ in terms of $z_{ij}$.

From $e_i = \sum_{k=1}^{M} A_{ik} e'_k$, $f_j = \sum_{i=1}^{N} B_{ij} f'_i$, and the linearity of the tensor product, we find

$e_i \otimes f_j = \left( \sum_{k=1}^{M} A_{ik} e'_k \right) \otimes \left( \sum_{i=1}^{N} B_{ij} f'_i \right) = \sum_{k=1}^{M} \sum_{i=1}^{N} A_{ik} B_{ij} (e'_k \otimes f'_i)$. So, $z = \sum_{i=1,j=1}^{M,N} z_{ij} (e_i \otimes f_j)$ implies that $z = \sum_{k=1,j=1}^{M,N} z'_{kj} (e'_k \otimes f'_j)$. Thus, $z'_{kj}$, which is the coefficient of $e'_k \otimes f'_j$ in $z$, is $z'_{kj} = \sum_{i=1}^{M} \sum_{j=1}^{N} A_{ik} B_{ij} z_{ij}$.

B. As discussed in class (notes pg 14), the vector space $\text{Hom}(V, W)$ has a basis set $\{\psi_{y_1}, \psi_{y_2}, ..., \psi_{y_M}\}$ where $\psi_y$ is the homomorphism for which $\psi_y(e_i) = f_j$ and $\psi_y(e_u) = 0$ for $u \neq i$.

With the new basis sets for $V$ and $W$, $\text{Hom}(V, W)$ has a basis set $\{\psi'_{y_1}, \psi'_{y_2}, ..., \psi'_{y_M}\}$, with $\psi'_{y_j}(e'_i) = f'_j$, and $\psi'_{y_j}(e'_u) = 0$ for $u \neq i$. In the original basis set, any $\phi$ in $\text{Hom}(V, W)$ can be written as $\phi = \sum_{i=1,j=1}^{M,N} \phi_{y_i} \psi_{y_j}$, for some $M \times N$ array of scalars $\phi_{y_i}$. The exercise is to express $\phi = \sum_{i=1,j=1}^{M,N} \phi_{y_i} \psi_{y_j}$ in terms of the new basis set, namely as a sum $\phi = \sum_{k=1,j=1}^{M,N} \phi'_{y_j} \psi'_{y_k}$. That is, find $\phi'_{y_k}$ in terms of $\phi_{y_i}$.
From $\varphi = \sum_{k,l}^{M N} \varphi'_{kl} \psi'_{kl}$ and $e_i = \sum_{k=1}^M A_k e'_k$, we find

$$\varphi(e_i) = \sum_{k,l}^{M N} \varphi'_{kl} \psi'_{kl}(\sum_{u=1}^N A_u e'_u) = \sum_{k,l}^{M N} \sum_{u=1}^N A_u \varphi'_{ku} \psi'_{kl}(e'_u) = \sum_{k,l}^{M N} A_k \varphi'_{kl} f'_l.$$ 

where the final equality uses $\psi'_k(e'_k) = f'_l$ and $\psi'_k(e'_u) = 0$ for $u \neq k$.

Now we need to make use of $f_j = \sum_{i=1}^N B_{ij} f'_j$. In the original basis set, $\varphi(e_i) = \varphi_0 f_j$. So

$$\varphi(e_i) = \sum_{j=1}^N \varphi_0 f_j = \sum_{j=1}^N \sum_{i=1}^N B_{ij} \varphi_0 f'_j.$$ 

Putting together the two equations for $\varphi(e_i)$ yields $\sum_{k,l}^{M N} A_k \varphi'_{kl} f'_l = \sum_{i=1}^N \varphi_0 B_{ij} f'_j$, or,

$$\sum_{i=1}^N f'_j \left( \sum_{k,l}^{M N} A_k \varphi'_{kl} - \sum_{j=1}^N B_{ij} \varphi_0 \right) = 0.$$ 

Because the $f'_j$ are a basis for $W$, they are linearly independent. Therefore the only way that the above equation can be satisfied is if each coefficient of $f'_j$ is zero. That is,

$$\sum_{k,l}^{M N} A_k \varphi'_{kl} = \sum_{j=1}^N B_{ij} \varphi_0,$$ 

for all $l$. Note that this is a system of linear equations in the $\varphi'_{kl}$. We can solve it if we know the inverse of the matrix $A$, namely, the quantities $A^{-1}_{sk}$ for which

$$\sum_{i=1}^N A^{-1}_{sk} A_{ik} = \begin{cases} 1, & s = k \\ 0, & s \neq k \end{cases}.$$ 

(Convince yourself that the existence of $A^{-1}$ is guaranteed if both the $e_i$ and the $e'_i$ are a basis.)

Finally, from $\sum_{k=1}^M A_k \varphi'_{ki} = \sum_{j=1}^N B_{ij} \varphi_0$, we calculate

$$\sum_{i=1}^M \sum_{k=1}^M A^{-1}_{sk} A_{ik} \varphi'_{ki} = \sum_{i=1}^M \sum_{j=1}^N A^{-1}_{sk} B_{ij} \varphi_0$$ 

and apply $\sum_{i=1}^M A^{-1}_{sk} A_{ik} = \begin{cases} 1, & s = k \\ 0, & s \neq k \end{cases}$ to find

$$\varphi'_{ki} = \sum_{i=1}^M \sum_{j=1}^N A^{-1}_{sk} B_{ij} \varphi_0.$$ 

The “big-picture” point (compare the circled equations) is that for $\text{Hom}(V, W)$, $A^{-1}$ is applied to the $V$-component of the basis, while for $V \otimes W$, $A$ is applied to the $V$-component of the basis. So, a change of basis affects $\text{Hom}(V, W)$ and $V \otimes W$ differently.

There are two interesting special cases. 

First, take $W = k$, so $\text{Hom}(V, W) = V^*$ and $V \otimes W = V$ (convince yourself of this!). This exercise thus shows that $V^*$ and $V$ transform differently.
Second, take $W = V$, $f = e$, and $B = A$ (so, also, $f' = e'$) The exercise shows how $Hom(V, V)$ changes when coordinates of $V$ are changed, namely, $\phi'_{ij} = \sum_{i'}^{I} \sum_{j'}^{J} A^{-1}_{ii'} A_{jj'} \phi_{i'j'}$. Or, as standard matrices, $\phi' = A^{-1} \phi A$.

Q2: Coordinate-independent (natural) isomorphisms of vector spaces.

A. The dual of the dual. Consider $V^{**} = Hom(V^*, k) = Hom(Hom(V, k), k)$. That is, $V^{**}$ contains elements $\Phi$ that are linear mappings from $V^*$ to $k$. In other words, for two elements $\phi_1$ and $\phi_2$ of $V^*$, $\Phi(a\phi_1 + b\phi_2) = a\Phi(\phi_1) + b\Phi(\phi_2)$, where addition here is interpreted in $V^*$.

Construct a homomorphism $M$ from $V$ to $V^{**}$. That is, for any element $w$ in $V$, construct an element $\Phi_w = M(w)$ in $V^{**}$. To do this, you will have to

(i) come up with a rule for how $\Phi_w$ acts on elements $\phi$ of $V^*$,
(ii) show that $\Phi_w$ is linear on $V^*$, namely, that $\Phi_w(a\phi_1 + b\phi_2) = a\Phi_w(\phi_1) + b\Phi_w(\phi_2)$,
(iii) show that the map $M$ from $w$ to $\Phi_w$ is linear on $V$, namely, that $M(qw_1 + rw_2) = qM(w_1) + rM(w_2)$. (Addition on the left is interpreted in $V$; addition on the right is interpreted in $V^{**}$). Equivalently, $\Phi_{qw_1 + rw_2} = q\Phi_{w_1} + r\Phi_{w_2}$.

(i) Define $\Phi_w(\phi) = \phi(w)$. The right-hand side exploits the fact that since $\phi$ is in $V^*$, it is a linear map on elements of $V$.

(ii) As follows:
$\Phi_w(a\phi_1 + b\phi_2) = (a\phi_1 + b\phi_2)(w)$ (because of how $\Phi_w$ is defined, right-hand-side ops are in $V^*$)
$a\phi_1(w) + b\phi_2(w) = a\Phi_w(\phi_1) + b\Phi_w(\phi_2)$ (because of how addition and scalar multiplication are defined in $V^*$)

(iii) To show $\Phi_{qw_1 + rw_2} = q\Phi_{w_1} + r\Phi_{w_2}$, which is a statement about $V^{**}$, we must show that for all $\phi$ is in $V^*$, that $\Phi_{qw_1 + rw_2}(\phi) = q\Phi_{w_1}(\phi) + r\Phi_{w_2}(\phi)$.

$\Phi_{qw_1 + rw_2}(\phi) = \phi(qw_1 + rw_2) = q\phi(w_1) + r\phi(w_2) = q\Phi_{w_1}(\phi) + r\Phi_{w_2}(\phi)$.

In the above, first and third equalities are the definition of $\Phi_w$; second equality is because $\phi$ is a homomorphism.

Comment. This means that every element of $V$ can be regarded as an element of $V^{**}$, and this correspondence does not depend on coordinates.

B. Dual homomorphisms. Consider elements $\Psi$ in $Hom(V, W)$. Construct a homomorphism $M$ from $Hom(V, W)$ to $Hom(W^*, V^*)$. That is, given a homomorphism $\Psi$ from $V$ to $W$, construct a homomorphism $\Psi' = M(\Psi)$ from $W^*$ to $V^*$.
Say $\Psi$ is in $\text{Hom}(V,W)$. Say $\xi$ is in $W^\ast$ (so $\xi(w)$ is an element of $k$). $\Psi^\ast$ has to map $\xi$ to an element of $V^\ast$, i.e., $\Psi^\ast(\xi)$ needs to be defined by how it maps vectors $v$ of $V$ to field elements. We therefore define $\left(\Psi^\ast(\xi)\right)(v) = \xi(\Psi(v))$. (Note that since $\Psi$ is in $\text{Hom}(V,W)$, then $\Psi(v)$ is an element of $W$, so $\xi$ can act on it to yield a field element.). Properties (ii) and (iii) are straightforward, and shown in a manner analogous to part A.

Comment. Iterating this argument, one can construct $\Psi^{**} = (\Psi^*)^*$, which is a homomorphism from $\text{Hom}(V,W)$ to $\text{Hom}(V^{**},W^{**})$. In Part A, we saw that every element of $V$ can be regarded as an element of $V^{**}$ (and similarly for $W$). Given this identification, one can readily show that $\Psi^{**} = \Psi$.

C. Find a coordinate-free correspondence between $(V \otimes W)^\ast$ and $\text{Hom}(V,W^\ast)$.

Say $B$ is an element of $(V \otimes W)^\ast$. The means that $B(v \otimes w)$ is an element of the field $k$, and this expression is linear in $v$ and $w$.

We need to find an element $U_B$ of $\text{Hom}(V,W^\ast)$ that we can naturally associate with $B$. That is, $U_B$ must be a linear map from vectors $v$ to elements in the dual of $W$. To define $U_B(v)$ in the dual of $W$, we must define how it carries out a linear map from elements $w$ in $W$ to the field $k$. So we take $\left(U_B(v)\right)(w) = B(v \otimes w)$.

D. Find a coordinate-free correspondence between $V \otimes W$ and $\text{Hom}(V^\ast,W)$.

Say $v \otimes w$ is in $V \otimes W$. We need to find a linear map from $v \otimes w$ to an element $\Phi = Z(v \otimes w)$ in $\text{Hom}(V^\ast,W)$. To define $\Phi$, we need to show how it maps any $\varphi$ in $V^\ast$ to elements of $W$. We therefore define $\Phi = Z(v \otimes w)$ as $\left(Z(v \otimes w)\right)(\varphi) = \varphi(v)w$, which makes use of the fact that since $\varphi$ is in $V^\ast$, it maps vectors $v$ to scalars.