Groups, Fields, and Vector Spaces

Homework #3 (2008) for pages 9-16 of notes

Consider a vector space $V$ (with elements $v, \ldots$) over a field $k$ (with elements $a, b, \ldots$), and the dual space of $V$, denoted $V^*$. That is, $V^* = \text{Hom}(V,k)$, and consists of all the homomorphisms from $V$ to the field $k$. For example, a typical element of $V^*$ is a linear mapping $\phi$ from $V$ to $k$, satisfying $\phi(av + bv) = a\phi(v) + b\phi(v')$. Recall that when $V$ is finite-dimensional, then $V^*$ is also finite-dimensional BUT there is no natural way to set up a mapping from elements of $V$ to elements of $V^*$. In other words, to express a linear correspondence between $V$ and $V^*$, one needs to choose coordinates.

The point of these problems is (Q1) to spell out another, somewhat more elaborate, example of this: i.e., a correspondence between vector spaces that depends on the choice of coordinates, and (Q2) to demonstrate the contrasting situation: different vector spaces for which it is possible to set up a natural correspondence, independent of coordinates.

Q1. Coordinate-dependent isomorphisms of vector spaces

Given:
Vector space $V$ (with elements $v, \ldots$) and a basis set $\{e_1, e_2, \ldots, e_M\}$
Vector space $W$ (with elements $w, \ldots$) and a basis set $\{f_1, f_2, \ldots, f_N\}$

We'll construct two vector spaces of dimension $M \times N$, $V \otimes W$ and $\text{Hom}(V,W)$. We will then see what happens to the coordinates in these vector spaces when we change basis sets in $V$ and $W$ to new basis sets, $\{e'_1, e'_2, \ldots, e'_M\}$ for $V$ and $\{f'_1, f'_2, \ldots, f'_N\}$ for $W$. The new and old basis sets are related by $e_i = \sum_{k=1}^M A_{ik} e'_k$ and $f_j = \sum_{l=1}^N B_{jl} f'_l$.

A. As discussed in class (notes pg 16), the vector space $V \otimes W$ has a basis set $\{e_i \otimes f_j, e_2 \otimes f_1, \ldots, e_M \otimes f_N\}$, i.e., any element $z$ of $V \otimes W$ can be written in coordinates as $z = \sum_{i=1, j=1}^{M,N} z_{ij} (e_i \otimes f_j)$, for some $M \times N$ array of scalars $z_{ij}$.

The exercise is to express $z = \sum_{i=1, j=1}^{M,N} z_{ij} (e_i \otimes f_j)$ in terms of the new basis set for $V \otimes W$, namely as a sum $z = \sum_{k=1, l=1}^{M,N} z'_{kl} (e'_k \otimes f'_l)$. That is, find $z'_{kl}$ in terms of $z_{ij}$.

B. As discussed in class (notes pg 14), the vector space $\text{Hom}(V,W)$ has a basis set $\{\psi_{u_1}, \psi_{u_2}, \ldots, \psi_{MN}\}$ where $\psi_y$ is the homomorphism for which $\psi_y(e_i) = f_j$ and $\psi_y(e_u) = 0$ for $u \neq i$. With the new basis sets for $V$ and $W$, $\text{Hom}(V,W)$ has a basis set $\{\psi'_{u_1}, \psi'_{u_2}, \ldots, \psi'_{MN}\}$, with $\psi'_y(e'_i) = f'_j$, and $\psi'_y(e'_u) = 0$ for $u \neq i$. In the original basis set, any $\phi$ in $\text{Hom}(V,W)$ can be written as $\phi = \sum_{i=1, j=1}^{M,N} \phi_{ij} \psi_y$, for some
$M \times N$ array of scalars $\varphi_{ij}$. The exercise is to express $\varphi = \sum_{i,j}^{M,N} \varphi_{ij} \psi_{ij}$ in terms of the new basis set, namely as a sum $\varphi = \sum_{i,j}^{k,l} \varphi'_{kl} \psi_{kl}$. That is, find $\varphi'_{ij}$ in terms of $\varphi_{ij}$.

Q2: Coordinate-independent (natural) isomorphisms of vector spaces.

A. The dual of the dual. Consider $V'' = \text{Hom}(V^*,k) = \text{Hom}(\text{Hom}(V,k),k)$. That is, $V''$ contains elements $\Phi$ that are linear mappings from $V^*$ to $k$. In other words, for two elements $\varphi_1$ and $\varphi_2$ of $V^*$, $\Phi$ satisfies $\Phi(a\varphi_1 + b\varphi_2) = a\Phi(\varphi_1) + b\Phi(\varphi_2)$, where addition here is interpreted in $V^*$.

Construct a homomorphism $M$ from $V$ to $V''$. That is, for any element $w$ in $V$, construct an element $\Phi_w = M(w)$ in $V''$. To do this, you will have to

(i) come up with a rule for how $\Phi_w$ acts on elements $\varphi$ of $V^*$,
(ii) show that $\Phi_w$ is linear on $V^*$, namely, that $\Phi_w(a\varphi_1 + b\varphi_2) = a\Phi_w(\varphi_1) + b\Phi_w(\varphi_2)$,
(iii) show that the map $M$ from $w$ to $\Phi_w$ is linear on $V$, namely, that $M(qw_1 + rw_2) = qM(w_1) + rM(w_2)$. (Addition on the left is interpreted in $V$; addition on the right is interpreted in $V''$. Equivalently, $\Phi_{qw_1 + rw_2} = q\Phi_{w_1} + r\Phi_{w_2}$.

B. Dual homomorphisms. Consider elements $\Psi$ in $\text{Hom}(V,W)$. Construct a homomorphism $M$ from $\text{Hom}(V,W)$ to $\text{Hom}(W^*,V^*)$. That is, given a homomorphism $\Psi$ from $V$ to $W$, construct a homomorphism $\Psi^* = M(\Psi)$ from $W^*$ to $V^*$.

C. Find a coordinate-free correspondence between $(V \otimes W)^*$ and $\text{Hom}(V,W^*)$.

D. Find a coordinate free-correspondence between $V \otimes W$ and $\text{Hom}(V^*,W)$. 