Groups, Fields, and Vector Spaces

Homework #3 (2008) for pages 9-16 of notes

Consider a vector space *V* (with elements *v*, …) over a field *k* (with elements *a*, *b*, …), and the dual space of *V*, (page 14)denoted V^*. That is, $V^* = Hom(V,k)$, and consists of all the homomorphisms from *V* to the field *k*. For example, a typical element of V^* is a linear mapping φ from *V* to *k*, satisfying $\varphi(av_1 + bv_2) = a\varphi(v_1) + b\varphi(v_2)$. Recall that when *V* is finite-dimensional, then V^* is also finite-dimensional *BUT* there is no natural way to set up a mapping from elements of *V* to elements of V^*. In other words, to express a linear correspondence between *V* and V^*, one needs to choose coordinates.

The point of these problems is (Q1) to spell out another, somewhat more elaborate, example of this: i.e., a correspondence between vector spaces that depends on the choice of coordinates, and (Q2) to demonstrate the contrasting situation: different vector spaces for which it is possible to set up a natural correspondence, independent of coordinates.

Q1. Coordinate-dependent isomorphisms of vector spaces

Given:
Vector space *V* (with elements *v*, …) and a basis set $\{e_1, e_2, \ldots, e_M\}$
Vector space *W* (with elements *w*, …) and a basis set $\{f_1, f_2, \ldots, f_N\}$
We'll construct two vector spaces of dimension $M \times N$, $V \otimes W$ and $Hom(V,W)$. We will then see what happens to the coordinates in these vector spaces when we change basis sets in *V* and *W*. to new basis sets, $\{e_1', e_2', \ldots, e_M'\}$ for V and $\{f_1', f_2', \ldots, f_N'\}$ for *W*. The new and old basis sets are related by $e_i = \sum_{k=1}^{M} A_{ik} e_k'$ and $f_j = \sum_{l=1}^{N} B_{jl} f_l'$.

A. As discussed in class (notes pg 16), the vector space $V \otimes W$ has a basis set $\{e_1 \otimes f_1, e_1 \otimes f_2, \ldots, e_2 \otimes f_1, \ldots, e_M \otimes f_N\}$, i.e., any element *z* of $V \otimes W$ can be written in coordinates as $z = \sum_{i=1, j=1}^{M,N} z_{ij} \left(e_i \otimes f_j \right)$, for some $M \times N$ array of scalars z_{ij}.

The exercise is to express $z = \sum_{i=1, j=1}^{M,N} z_{ij} \left(e_i \otimes f_j \right)$ in terms of the new basis set for $V \otimes W$, namely as a sum $z = \sum_{k=1, l=1}^{M,N} z_{kl}' \left(e_k' \otimes f_l' \right)$. That is, find z_{kl}' in terms of z_{ij}.

B. As discussed in class (notes pg 14), the vector space $Hom(V,W)$ has a basis set $\{\psi_{11}, \psi_{12}, \ldots, \psi_{MN}\}$ where ψ_{ij} is the homomorphism for which $\psi_{ij}(e_i) = f_j$ and $\psi_{ij}(e_u) = 0$ for $u \neq i$. With the new basis sets for *V* and *W*, $Hom(V,W)$ has a basis set $\{\psi_{11}', \psi_{12}', \ldots, \psi_{MN}'\}$, with $\psi_{ij}'(e_i') = f_j'$, and $\psi_{ij}'(e_u') = 0$ for $u \neq i$. In the original basis set, any φ in $Hom(V,W)$ can be written as $\varphi = \sum_{i=1, j=1}^{M,N} \varphi_{ij} \psi_{ij}$, for some

$M \times N$ array of scalars φ_{ij}. The exercise is to express $\varphi = \sum_{i=1, j=1}^{M,N} \varphi_{ij}\psi_{ij}$ in terms of the new basis set, namely as a sum $\varphi = \sum_{k=1,l=1}^{M,N} \varphi'_{kl}\psi'_{kl}$. That is, find φ'_{kl} in terms of φ_{ij}.

Q2: Coordinate-independent (natural) isomorphisms of vector spaces.

A. The dual of the dual. Consider $V^{**} = Hom(V^*, k) = Hom(Hom(V,k),k)$. That is, V^{**} contains elements Φ that are linear mappings from V^* to *k*. In other words, for two elements φ_1 and φ_2 of V^*, Φ satisfies $\Phi(a\varphi_1 + b\varphi_2) = a\Phi(\varphi_1) + b\Phi(\varphi_2)$, where addition here is interpreted in V^*.

Construct a homomorphism *M* from *V* to V^{**}. That is, for any element *w* in *V*, construct an element $\Phi_w = M(w)$ in V^{**}. To do this, you will have to
(i) come up with a rule for how Φ_w acts on elements φ of V^*,
(ii) show that Φ_w is linear on V^*, namely, that $\Phi_w(a\varphi_1 + b\varphi_2) = a\Phi_w(\varphi_1) + b\Phi_w(\varphi_2)$,
(iii) show that the map *M* from *w* to Φ_w is linear on *V*, namely, that $M(qw_1 + rw_2) = qM(w_1) + rM(w_2)$. (Addition on the left is interpreted in *V*; addition on the right is interpreted in V^{**}. Equivalently, $\Phi_{qw_1+rw_2} = q\Phi_{w_1} + r\Phi_{w_2}$.

B. Dual homomorphisms. Consider elements Ψ in $Hom(V,W)$. Construct a homomorphism *M* from $Hom(V,W)$ to $Hom(W^*,V^*)$. That is, given a homomorphism Ψ from *V* to *W*, construct a homomorphsm $\Psi^* = M(\Psi)$ from W^* to V^*.

C. Find a coordinate-free correspondence between $(V \otimes W)^*$ and $Hom(V,W^*)$.

D. Find a coordinate free-correspondence between $V \otimes W$ and $Hom(V^*,W)$.