Linear Systems Theory

Goal: provide a concise way of characterizing

\[ s(t) \xrightarrow{F} r(t) \]

so that one can (a) describe it,
(b) test models for it
(c) make guesses as to what's inside
(d) make use of partial knowledge

Examples: electric circuit, physiology

- VOR - rotate head, observe eye movement
- volume homeostasis - apply volume load
- membrane - unmeasured
- apply current, measure voltage
- light to voltage
- neural activity to blood flow

For general \( F \), this is intractable.
So we make two assumptions:
- translation - invariance
- linearity
- (and causality) (Post-processing filters need not be causal.)
Translation: if \[ F(s) \] (t) = r(t) \]
and \[ s' = D_\gamma (s) \quad (\text{i.e., } s'(t) = s(t+\gamma)) \]
then \[ [F(s')] (t) = r(t+\gamma) \]
\text{i.e., } F D_\gamma = D_\gamma F

Linearity: \[ [F(s_1+s_2)] (t) = [F(s_1)] (t) + [F(s_2)] (t) \]
and \[ (F (2s)) (t) = 2 [F(s)] (t) \]

Linearity is always an approximation.
The theory does not fit to non linear transformable in part.

Linearity allows us to consider \( F \) to be an element of \( \text{Hom}(V, V) \) where \( V \) is the \( \text{V. S. of functions of time.} \)

But if \( F \) is a physical system, then how to interpret \([F(s)] (t)\) for \( s \) complex-valued?

Ans: linearity, \( s(t) = a(t) + i b(t) \)
so \( F(s) (t) = (F(a)) (t) + i [F(b)] (t) \)
Linearity gives us an intuitive way of characterizing $F$.

$S(t) = \int$ [diagram]

Parcel it into small intervals

Linearity means that we can consider each interval independently.

$S(t) = \int$ [diagram]

Response at time $t = \text{sum of contributions of } S \text{ at all times in past } t$. Say a unit pulse at time 0 leads to a response $f(\tau)$. Equivalently, a unit pulse at time $t-\tau$ leads to a response $f(\tau)$ at time $t$.

$r(t) = \int_{\tau=0}^{\infty} f(\tau)S(t-\tau)\,d\tau$. $f$ is the "impulse response."
What's the problem with this description?

1. Estimating \( f(\tau) \). You need to work where
   \( \text{infinity is most likely to break down} \).

2. Combining systems. Parallel is easy.

Consider

\[
\begin{aligned}
    s(t) & \xrightarrow{F} q(t) \xrightarrow{G} r(t) \\
    H & \\
\end{aligned}
\]

If you know \( f(\tau) = q(\tau) \), what is \( h(\tau) \)?

Say \( F(s) = q, \quad G(q) = r \). \[ r(t) = \int q(\tau) q(t - \tau) d\tau \]

\[ q(t) = \int f(\tau') s(t - \tau') d\tau' \]

So \( r(t) = \int \int q(\tau) f(\tau') s(t - \tau - \tau') d\tau' d\tau \)

\[ u = \tau + \tau' \]

\[ = \int \int q(\tau) f(u - \tau) s(t - u) d\tau' du \]

\[ = \int h(u) s(t - u) du \quad \text{for} \quad h(u) = \int g(\tau)f(u - \tau) d\tau. \]
So, for systems in series,

\[ h(u) = \int g(\tau) f(u-\tau) d\tau \text{, or } h = f * g \]

It's not so easy to intuit what this means on, say, a given \( h \) and \( f \), to solve for \( g \).

And consider:

\[ H = \]

or more complex networks.

What we've done above is to describe \( F \) in terms of the basis set of time functions:

\[ \int \]

These are the delta-function.

\( \delta(t,\tau) \)

\( Dp \) permutes them. But we know there is another basis set for which \( Dp \) multiple these, by Euler.

So now let's change basis.

In the new basis set, \( \text{e}^{i\omega t} \), \( F \) must act as a multiple of \( \text{e}^{i\omega t} \).

i.e., for \( s(t) = \text{e}^{i\omega t} \),

then \( [F(s)](t) = \text{a multiple of } \text{e}^{i\omega t} \).
Let's find the multiplicative constant \( \hat{f}(\omega) \), the "transfer function" of \( f \).

If \( s(t) = e^{i \omega t} \), then

\[
[P(s)](t) = \int f(\tau) s(t-\tau) d\tau
\]

\[
= \int f(\tau) e^{i \omega (t-\tau)} d\tau
\]

\[
= e^{i \omega t} \int f(\tau) e^{-i \omega \tau} d\tau
\]

So,

\[
\hat{f}(\omega) = \int e^{-i \omega \tau} f(\tau) d\tau.
\]

Interpret \( \hat{f}(\omega) \) in terms of real signals. Say \( f(t) = |f(\omega)| e^{i \phi(\omega)} \).

\[
s(t) = e^{i \omega t} = \cos(\omega t + i \sin(\omega t))
\]

\[
r(t) = |f(\omega)| e^{i \phi(\omega)} s(t)
\]

\[
= |f(\omega)| \left( \cos \phi(\omega) \cos(\omega t) + i \sin \phi(\omega) \sin(\omega t) \right)
\]

\[
= |f(\omega)| \left[ \cos(\phi(\omega) \cos(\omega t) - \sin \phi(\omega) \sin(\omega t)) \right] + i \left[ \sin \phi(\omega) \cos(\omega t) + \cos \phi(\omega) \sin(\omega t) \right]
\]

\[
= |f(\omega)| \left[ \cos(\omega t + \phi(\omega)) + i \sin(\omega t + \phi(\omega)) \right]
\]

Amplitude \( |f(\omega)| \), Phase shift \( \phi(\omega) \).
Does the transfer function help with the "problems" on pg. 4?

@ measurement. Use $\triangle \cos(\omega t)$.

@ composition

$$s(t) \xrightarrow{F} F \xrightarrow{G} G \xrightarrow{r(t)}$$

Put in $e^{i\omega t} = s(t)$. Then $q(t) = F(w) e^{i\omega t}$.

Next, since $G$ is linear, $r(t) = G(w) \left[ F(w) e^{i\omega t} \right]$.

$$= G(w) \hat{F}(w) e^{i\omega t}$$

So $\hat{r}(w) = \hat{F}(w) \hat{G}(w)$. "The Convolution Thm."

What if we want to calculate how $F$ responds to arbitrary $s$, but only have $\hat{F}(w)$? Is there a way to find $F(t)$ from $\hat{F}(w)$?

We need to write $s(t) = \int e^{i\omega t} \hat{S}(\omega) d\omega$

Since, then we could calculate $f(t) = [F(s)](t)$ from its response to $e^{i\omega t}$. 
"It turns out" that
\[ \hat{F}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \]

so the response to a \( S \)-factor is
\[ f(t) = \left[ \hat{F}(t) \right](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega. \]

\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt \]
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega \]

Fourier Transform Pairs

A way to see this - pp 4-5 of 2004 notes FAPP.
An example of combining elements in a more complex way: feedback

\[ s(t) \rightarrow F \rightarrow r(t) \]

\[ a(t) = s(t) + k b(t) \]

\[ b(t) = g(\omega) \hat{h}(\omega) e^{j\omega t} \]

\[ \hat{h}(\omega) e^{j\omega t} = f(\omega) e^{j\omega t} (1 + k \hat{g}(\omega) \hat{h}(\omega)) \]

\[ \hat{h}(\omega) = \frac{f(\omega)}{1 - k \hat{g}(\omega) \hat{h}(\omega)} \]
Apply Ohm's law, measure voltage - in a network of R's - C's.

\[ V = IR \]

\[ \hat{f}(\omega) = \frac{1}{j\omega R} \]

\[ f(t) = \frac{1}{\sqrt{2}} \]

\[ Q = CV \]

\[ I = \frac{dQ}{dt} = C \frac{dV}{dt} \]

\[ \frac{dV}{dt} = \frac{1}{C} I \]

\[ I = e^{j\omega t} \]

\[ V = \hat{f}(\omega) e^{j\omega t} \]

\[ \Rightarrow \frac{dV}{dt} = j\omega e^{j\omega t} \hat{f}(\omega) \]

\[ \hat{f}(\omega) j\omega e^{j\omega t} = \frac{1}{C} e^{j\omega t} \]

\[ \hat{f}(\omega) = \frac{1}{j\omega C} \]

R's + C's will always lead to algebraic combination & \( R = \frac{1}{j\omega C} \), i.e.,

rational expressions in \( \omega \).