Linear Systems Theory

Homework #1 (2008) Answers

Q1. Some basic properties of Fourier transforms pairs,

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{1}$$

and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t}d\omega . \tag{2}$$

A. Let $g(t) = e^{iat} f(t)$. *Find* $\tilde{g}(\omega)$ *in terms of* $\tilde{f}(\omega)$.
Beginning with (1),

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{iat}e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{-i(\omega-a)t}dt = \tilde{f}(\omega-a) .$$

B. Let $g(t) = kf(kt)$. *Find* $\tilde{g}(\omega)$ *in terms of* $\tilde{f}(\omega)$.
Beginning with (1),

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} kf(kt)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(kt)e^{-i\frac{\omega}{k}kt}d(kt) = \tilde{f}(\frac{\omega}{k}) .$$

C. Let $g(t) = \frac{df(t)}{dt}$. *Find* $\tilde{g}(\omega)$ *in terms of* $\tilde{f}(\omega)$.
Beginning with (2),

$$\frac{df}{dt} = \frac{1}{2\pi}\frac{d}{dt}\left(\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t}d\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\frac{d}{dt}e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega\tilde{f}(\omega)e^{i\omega t}d\omega .$$

So $g(t) = \frac{df(t)}{dt}$ and $i\omega\tilde{f}(\omega)$ are Fourier transform pairs, so $\tilde{g}(\omega) = i\omega\tilde{f}(\omega)$.

D. Find $\int_{-\infty}^{\infty} f(t)dt$ *in terms of* $\tilde{f}(\omega)$.

Set $\omega = 0$ in (1) to find $\tilde{f}(0) = \int_{-\infty}^{\infty} f(t)dt$.

E. Find $\int_{-\infty}^{\infty} t^m f(t)dt$ *in terms of* $\tilde{f}(\omega)$.

Start with (1) and differentiate both sides with respect to ω:

$$\frac{d\tilde{f}}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} (-it)f(t)e^{-i\omega t}dt .$$

Repeating *m* times: $\dfrac{d^m\tilde{f}}{d\omega^m}=\dfrac{d}{d\omega}\int\limits_{-\infty}^{\infty}(-it)^{m-1}f(t)e^{-i\omega t}dt=\int\limits_{-\infty}^{\infty}(-it)^m f(t)e^{-i\omega t}dt$.

Evaluate at $\omega=0$: $\dfrac{d^m\tilde{f}}{d\omega^m}(0)=\int\limits_{-\infty}^{\infty}(-it)^m f(t)dt$, so, $i^m\dfrac{d^m\tilde{f}}{d\omega^m}(0)=\int\limits_{-\infty}^{\infty}t^m f(t)dt$.

(Note relationship to part C).

F. Show that if $f(t)=f(-t)$ *, then* $\tilde{f}(\omega)$ *is real.*

Start with (1) and calculate the complex conjugate of $\tilde{f}(\omega)$:

$\overline{\tilde{f}(\omega)}=\overline{\int\limits_{-\infty}^{\infty}f(t)e^{-i\omega t}dt}=\int\limits_{-\infty}^{\infty}f(t)e^{i\omega t}dt=\int\limits_{-\infty}^{\infty}f(-u)e^{-i\omega u}du=\int\limits_{-\infty}^{\infty}f(u)e^{-i\omega u}du=\tilde{f}(\omega)$, where we have substituted $u=-t$. Since $\tilde{f}(\omega)$ is equal to its complex conjugate, it must be real.

Q2: Smoothing and averaging filters

A. Boxcar average. Define $f_{boxcar}(t)=\begin{cases}\dfrac{1}{L},|t|\le L/2\\ 0, otherwise\end{cases}$. *This replaces s by its average over a window of length L. Find the corresponding transfer function* $\tilde{f}_{boxcar}(\omega)$.

$$\tilde{f}_{boxcar}(\omega)=\int\limits_{-\infty}^{\infty}f_{boxcar}(t)e^{-i\omega t}dt=\frac{1}{L}\int\limits_{-L/2}^{L/2}e^{-i\omega t}dt=\frac{1}{L}\frac{1}{-i\omega}e^{-i\omega t}\Big|_{-L/2}^{L/2}=\frac{1}{2}\frac{e^{i\omega L/2}-e^{-i\omega L/2}}{i\omega L/2}=\frac{\sin(\omega L/2)}{\omega L/2}.$$

Note that $\mathrm{sinc}(u)\equiv\dfrac{\sin(u)}{u}$ is continuous at 0, and has a limit of 1 as *u* approaches 0. It has zeros at $u=\pm k\pi$ (integer $k\neq 0$), and maxima approximately halfway in between.

B. Triangular average. Define $f_{triangle}(t)=\begin{cases}\dfrac{(1-|t|/L)}{L},|t|\le L\\ 0, otherwise\end{cases}$. *This replaces s by its average over a window of length L but weights the central values more heavily Find the corresponding transfer function* $\tilde{f}_{triangle}(\omega)$. *Relate the answer to part A.*

$$\tilde{f}_{triangle}(\omega) = \int_{-\infty}^{\infty} f_{triangle}(t)e^{-i\omega t}dt = \frac{1}{L^2}\int_{-L}^{L}(L-|t|)e^{-i\omega t}dt = \frac{2}{L^2}\mathrm{Re}\left\{\int_0^L (L-t)e^{-i\omega t}dt\right\}$$, where the final equality holds by a symmetry argument: for negative values of *t*, the integrand has the same real part but the negative of the imaginary part. Then,

$$\tilde{f}_{triangle}(\omega) = \frac{2}{L^2}\mathrm{Re}\left\{\int_0^L (L-t)e^{-i\omega t}dt\right\} = \frac{2}{L^2}\int_0^L (L-t)\cos(\omega t)dt = \frac{2}{L^2}\left(L\frac{\sin\omega t}{\omega} - t\frac{\sin\omega t}{\omega} - \frac{\cos\omega t}{\omega^2}\right)\Bigg|_0^L$$

$$= \frac{2}{L^2}\left(L\frac{\sin\omega L}{\omega} - L\frac{\sin\omega L}{\omega} - \frac{\cos\omega L}{\omega^2} + \frac{1}{\omega^2}\right) = \frac{2}{L^2}\left(\frac{1-\cos\omega L}{\omega^2}\right) = \frac{2}{L^2}\left(\frac{2\sin^2(\omega L/2)}{\omega^2}\right) = \left(\frac{\sin(\omega L/2)}{\omega L/2}\right)^2$$

.

Note that $\tilde{f}_{triangle}(\omega) = \left(\tilde{f}_{boxcar}(\omega)\right)^2$. This corresponds to the fact that $f_{triangle}(t) = f_{boxcar}(t) * f_{boxcar}(t)$ via the convolution theorem. The fact that $f_{triangle}(t) = f_{boxcar}(t) * f_{boxcar}(t)$ can be shown via direct calculation or via a simple argument that this convolution must be symmetric, must be positive, must peak at 0, must be 0 for $|t| > L$, and must behave linearly between its peak at 0 and its 0-value at *L*.

C. Cosine bell. $f_{bell}(t) = \begin{cases} \dfrac{1+\cos(\pi t/L)}{2L}, |t| \le L \\ 0, otherwise \end{cases}$. *Find the corresponding transfer function* $\tilde{f}_{bell}(\omega)$.

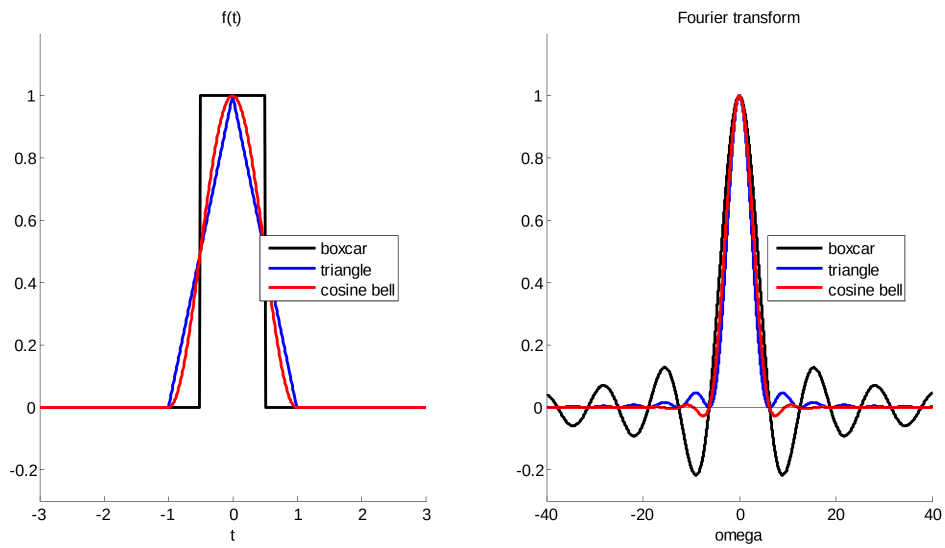
$$\tilde{f}_{bell}(\omega) = \int_{-\infty}^{\infty} f_{bell}(t)e^{-i\omega t}dt = \frac{1}{2L}\int_{-L}^{L}\left(1+\frac{e^{-\pi it/L}+e^{\pi it/L}}{2}\right)e^{-i\omega t}dt.$$

Note that the first term, $\frac{1}{2L}\int_{-L}^{L} e^{-i\omega t}dt$, corresponds to the boxcar kernel of Q2A, but spread out by a factor of 2, i.e., $\frac{1}{2}f_{boxcar}(t/2)$. So, using the result of Q1B with $k = 1/2$, its Fourier transform is $\tilde{f}_{boxcar}(\frac{\omega}{k})$, or, $\mathrm{sinc}(\omega L)$. For the second and third terms, use the result of Q1A, with $a = \pm\pi/L$. So,

$$\tilde{f}_{bell}(\omega) = \tilde{f}_{boxcar}(2\omega) + \frac{1}{2}\left(\tilde{f}_{boxcar}(2\omega+\pi/L) + \tilde{f}_{boxcar}(2\omega-\pi/L)\right)$$
$$= \mathrm{sinc}(\omega L) + \frac{1}{2}\left(\mathrm{sinc}(\omega L+\pi) + \mathrm{sinc}(\omega L-\pi)\right).$$

D. Which of the above would you want to use?

All have their first zero-crossing at $\omega = 2\pi$, but the cosine bell has the least "ringing". See graph below.


f(t)
Fourier transform
boxcar
triangle
cosine bell
t
omega
1
0.8
0.6
0.4
0.2
0
-0.2
-3
-2
-1
0
1
2
3
-40
-20
0
20
40