Q1. Some basic properties of Fourier transforms pairs,

\[
\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt
\]
and

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t} d\omega .
\]

A. Let \( g(t) = e^{i\omega t} f(t) \). Find \( \tilde{g}(\omega) \) in terms of \( \tilde{f}(\omega) \).

B. Let \( g(t) = kf(kt) \). Find \( \tilde{g}(\omega) \) in terms of \( \tilde{f}(\omega) \).

C. Let \( g(t) = \frac{df(t)}{dt} \). Find \( \tilde{g}(\omega) \) in terms of \( \tilde{f}(\omega) \).

D. Find \( \int_{-\infty}^{\infty} f(t)dt \) in terms of \( \tilde{f}(\omega) \).

E. Find \( \int_{-\infty}^{\infty} t^n f(t)dt \) in terms of \( \tilde{f}(\omega) \).

F. Show that if \( f(t) = f(-t) \), then \( \tilde{f}(\omega) \) is real.

Q2: Smoothing and averaging filters

Here, we view a smoothing and averaging filter \( F \) as a linear transformation on unprocessed signals \( s(t) \), to produce a processed signal \( r(t) = [F(s)](t) \). But since the transformation can be applied after all data are collected, the “impulse response” function \( f(t) \) need not be causal. That is,

\[
r(t) = \int_{-\infty}^{\infty} f(\tau)s(t-\tau)d\tau
\]
where \( f(t) \) can be nonzero for both negative and positive times. There is no change in how the transfer function is defined, namely,

\[
\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt
\]
A. Boxcar average. Define $f_{\text{boxcar}}(t) = \begin{cases} \frac{1}{L} & |t| \leq L/2 \\ 0, & \text{otherwise} \end{cases}$. This replaces $s$ by its average over a window of length $L$. Find the corresponding transfer function $\tilde{f}_{\text{boxcar}}(\omega)$.

B. Triangular average. Define $f_{\text{triangle}}(t) = \begin{cases} \frac{(1-|t|/L)}{L} & |t| \leq L \\ 0, & \text{otherwise} \end{cases}$. This replaces $s$ by an average over a window of length $L$ but weights the central values more heavily. Find the corresponding transfer function $\tilde{f}_{\text{triangle}}(\omega)$. Relate the answer to part A.

C. Cosine bell. $f_{\text{bell}}(t) = \begin{cases} \frac{1+\cos(\pi t/L)}{2L} & |t| \leq L \\ 0, & \text{otherwise} \end{cases}$. Find the corresponding transfer function $\tilde{f}_{\text{bell}}(\omega)$.

D. Which of the above would you want to use?