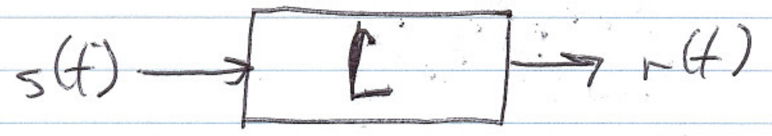
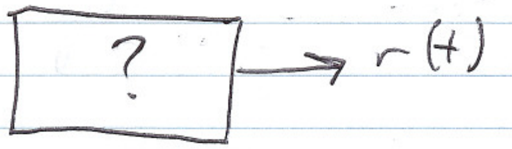
① Fourier Analysis – Applications: Noise + Variability

A cluster of settings + problems:

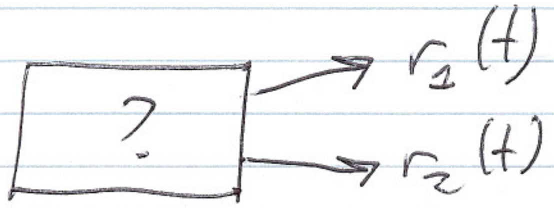


but repeating the experiment does not lead to the same response ("Noise")

- system noise
- measurement noise



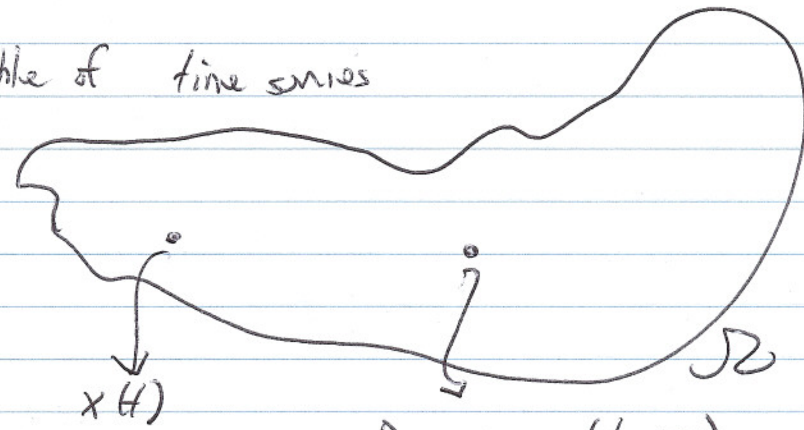
How to characterize r? Can we say anything about the system?



Is there a relationship between r_1 + r_2?

In each case, we need to make use of observations made at different times. But we're thinking of this as repeated observations of samples drawn from an ensemble.

An ensemble of time series



Characterize $p(x)$ $\qquad$ $D_\tau(x) = x(t+\tau)$

Typically, we "sample the ensemble" by waiting, resampling again.

So we need:

(a) Stationarity. $\qquad p(x(t)) = p(D_\tau(x)(t))$

(b) Ergodicity $\qquad \langle \quad \rangle_\Omega = \langle \quad \rangle_\tau$

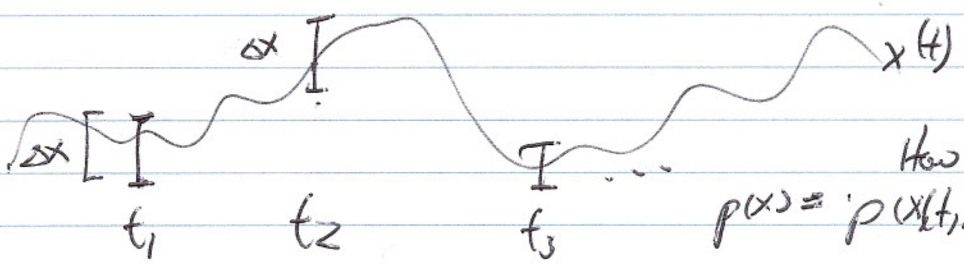
ensemble-average = time-average

(c) Limited correlations in time:

for sufficiently long T, $x(t)$ and $x(t+T)$ are independent.

Note: How to define $p(x)$, when x is a time series?

Easy to define $p(x(t_i))$, for t_i a single time point



How to avoid

$p(x) = p(x(t_1))\,p(x(t_2))\,p(x(t_3)) \ldots = 0$?

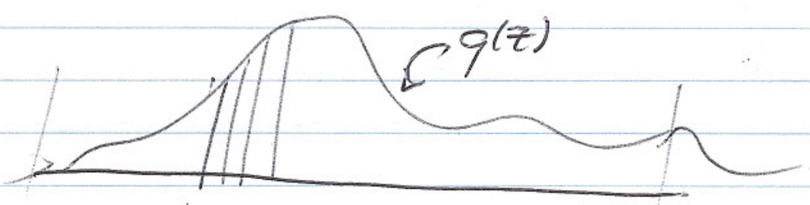
(3)

Probability that $x(t)$ jumps through hoops of width Δx, at N times, is (for small Δx) proportional to $(\Delta x)^N$.

As $N \Rightarrow \infty$, this goes to 0, but, this is not a problem...

it is analogous to what happens when you discretize a probability distribution, ~~but~~ (discrete bins of size Δx), but here, the discretization has N dimensions, & $N \Rightarrow \infty$.

E.g.: say z is an ordinary (scalar) random variable:



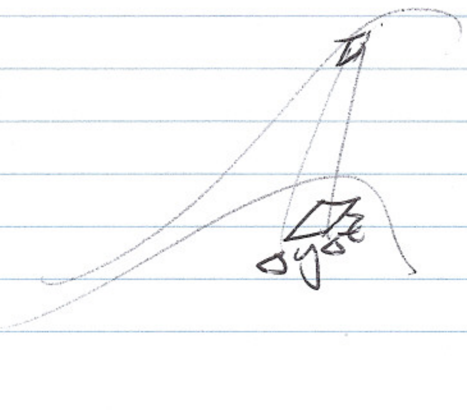
$q(z)$ is the probability dist. associated $\bar{c}$ z.

$q(z_0)\Delta z$ is prob that z is between z_0 & $z_0 + \Delta z$.

Average $\langle z \rangle = \lim\limits_{\Delta z \Rightarrow 0} \sum_i z\, q(z_i)\Delta z = \int_{-\infty}^{\infty} z\, q(z)\, dz$

& in general $\langle f(z) \rangle = \int_{-\infty}^{\infty} f(z)\, q(z)\, dz$.

In 2-d,



$\langle f \rangle = \int f(y,z)\, q(y,z)\, dy\, dz$

"etc."

(4)

One way to understand $p(x)$ is from its moments (mean, variance, covariance, etc.)

$\langle x(\tau)\rangle$ must be independent of τ (translation-invariance)

so it is not very useful. Instead can always assume $\langle x(\tau)\rangle = 0$

So let's look at second moments, e.g.

$\langle x(\tau_1)\, x(\tau_2)\rangle$ and $\langle x(\tau_1)^2\rangle$.

Write $C_x(\tau_1, \tau_2) = \langle x(\tau_1)\, x(\tau_2)\rangle$

($\langle x(\tau)\rangle \neq 0$: $C_x(\tau_1, \tau_2) = \langle (x(\tau_1) - \langle x(\tau)\rangle)(x(\tau_2) - \langle x(\tau)\rangle)\rangle$)

$\langle x(\tau_1 + t)\, x(\tau_2 + t)\rangle = \langle x(\tau_1)\, x(\tau_2)\rangle$ (translation-invariance)

so $C_x(\tau_1 + t, \tau_2 + t) = C_x(\tau_1, \tau_2) \equiv C_x(\tau_1 - \tau_2)$

This is the "autocovariance". Note $C_x(\tau) = C_x(-\tau)$

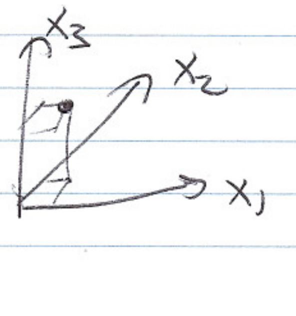
and

$C_x(\tau) \leq C_x(0)$

Autocorrelation $= C_x(\tau)/C_x(0)$

Geometrically: Sample $x(t)$ at $\tau_1, \cdots, \tau_n$.

So $x(t)$ represented by an n-vector, $x(t_i)$ $(i = 1, \cdots, n)$

$x \longrightarrow$ 

, distribution of x is a cloud. What is its shape?

(5)

With n sample, there are $\frac{n(n-1)}{2}$ covariances,

some of them must be equal ($\langle X_n X_{n+k} \rangle$)

and they are not independent:

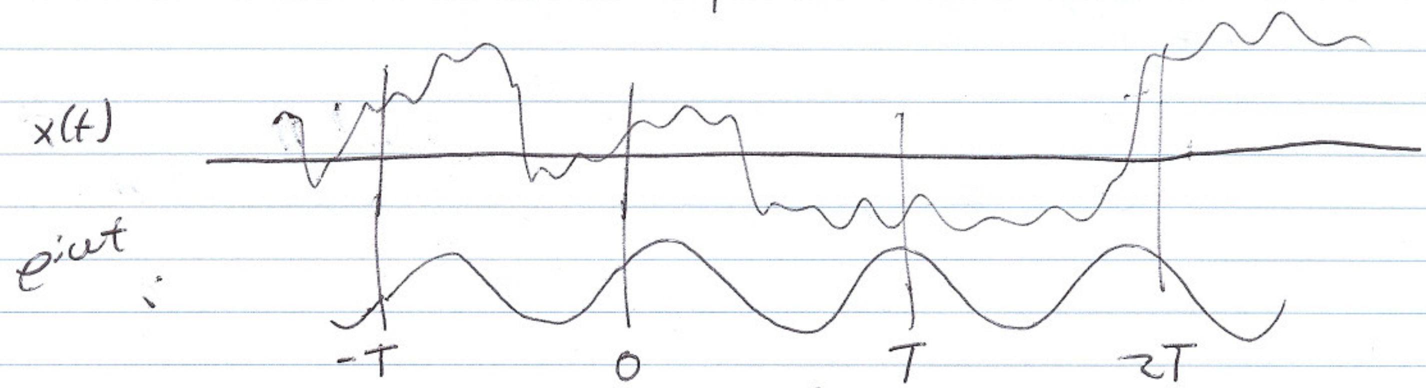
large $\langle X_2 X_3 \rangle$ and large $\langle X_3 X_5 \rangle$

$\Longrightarrow$ probably large $\langle X_2 X_5 \rangle$.

So how do we efficiently estimate $C_X(t)$, or
how do we put error bars on it?

Change basis! Characterize $\tilde{X}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$

But $\tilde{X}(\omega)$ is ill-defined!



$$\int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt = \sum_{N=-\infty}^{\infty} \int_{NT}^{(N+1)T} x(t) e^{-i\omega t} dt = \sum_{N=-\infty}^{\infty} F_N$$

where F_N = a "Fourier estimate" at freq. ω, length T,
start time NT

⑥

$F_N = F(x; \omega, T, NT)$ where

$\uparrow$ signal $\uparrow$ freq $\uparrow$ dur $\uparrow$ start

$$F(x; \omega, T, T_0) = \int_{T_0}^{T_0+T} x(t)\, e^{-i\omega t}\, dt.$$

How is F distributed?

Put $u = t + T_0$. $F(x; \omega, T, T_0) = \int_0^{T_0} x(u - T_0)\, e^{-i\omega(u-T_0)}\, du$

$$= e^{i\omega T_0} \int_0^T x(u-T_0)\, e^{-i\omega u}\, du$$

$$= e^{i\omega T_0} \int_0^T [D_{-T_0}(x)](u)\, e^{-i\omega u}\, du$$

$$= e^{i\omega T_0} F(D_{-T_0}(x); \omega, T, 0)$$

But $F(D_T(x); \omega, T, 0)$ must be distrib like $F(x; \omega, T, 0)$

(translation-invariance),

AND

$F(x; \omega, T, T_0)$ must have a distrib indep of T_0.

So $F(x; \omega, T, T_0)$ is circularly symmetric in the complex plane AND indep of T_0.

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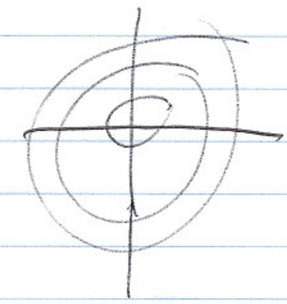
$$I_{KT} = \int_0^{KT} x(t)\, e^{-i\omega t}\, dt = \sum_{N=0}^{K-1} F_N .$$

each F_N is an fourier estimate, beginning at time NT, of duration T. It has a distribution that is circularly symmetric in the complex plane.



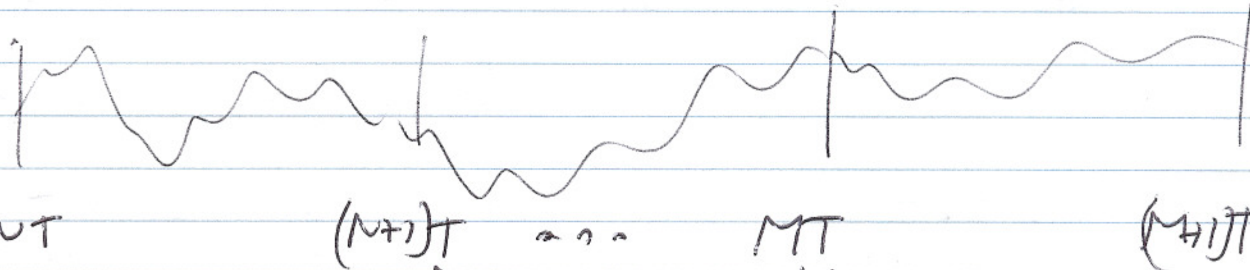
So $I_{KT} = 0$.

What about $\langle |I_{KT}|^2 \rangle$?

$$\langle |I_{KT}|^2 \rangle = \langle |\sum_{N=0}^{K-1} F_N|^2 \rangle = \langle \sum_{N=0}^{K-1} F_N \sum_{M=0}^{K-1} \overline{F_M} \rangle$$

$$= \sum_{N,M=0}^{K-1} \langle F_N \overline{F_M} \rangle .$$

Two kinds of terms: $N=M$, $N \neq M$.



For long T, F_N & F_M are uncorrelated.

So, for long T, cross-terms → 0 and

$$\langle |I_{KT}|^2 \rangle = \sum_{N=0}^{K-1} |F_N|^2 = K \langle |F(x; \omega, T, 0)|^2 \rangle$$

$$I_{KT} = \int_0^{KT} x(t)\, e^{-i\omega t}\, dt$$

has a variance proportional to K, provided T is large enough. I.e.,

$$I_L = \int_0^L x(t)\, e^{-i\omega t}\, dt$$ has

a variance proportional to L (& a mean of 0) for L sufficiently large.

$$P_x(\omega) = \lim_{L \to \infty} \frac{1}{L} \langle |I_L(\omega)|^2 \rangle$$

This is the "power spectrum"

$$P_x(\omega) = \lim_{T \to \infty} \frac{1}{T} \left\langle \left| \int_0^T x(t)\, e^{-i\omega t}\, dt \right|^2 \right\rangle_\Omega$$

Units of p.s. = $X^2 \cdot$ Time

Replacing $\langle\ \rangle_\Omega$ by $\langle\ \rangle_{T_0}$,

$$P_x(\omega) = \lim_{\substack{T \to \infty \\ N \to \infty}} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{T} |F(x; \omega, T, nT)|^2$$

Practical difficulties: Can't make $T \to \infty$.

With a finite amount of data, length L, must have $NT \leq L$, so you can't have

$T \to \infty$

$N \to \infty$.

Large T: better est. of $F(x; \omega, T, nT)$ but fewer examples

⑨

What about the "off-diagonal" terms, like

☆ $$\langle F(x; \omega_1, L, T_1)\, F(x; \omega_2, L, T_2) \rangle \; ?$$

If start times of intervals $[T_1, T_1+L]$, $[T_2, T_2+L]$ are well-separated, then (c) $\Rightarrow$ independence.

If not well-separated: say $T_1 = T_2 = T$

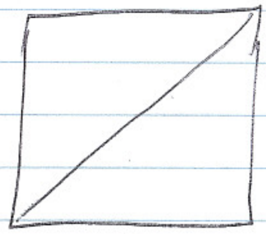
$$\left\langle \int_T^{L+T} x(t)\, e^{-i\omega_1 t}\, dt \int_T^{L+T} x(t)\, e^{-i\omega_2 t}\, dt \right\rangle$$

$$= \left\langle \int_T^{L+T}\int_T^{L+T} x(t_1)x(t_2)\, e^{-i\omega_1 t_1 - i\omega_2 t_2}\, dt_1\, dt_2 \right\rangle$$

$$= \int_T^{L+T}\int_T^{L+T} \langle x(t_1)x(t_2)\rangle\, e^{-i\omega_1(t_1-t_2) - i\omega_1 t_2 - i\omega_2 t_2}\, dt_1\, dt_2$$

$$= \int_T^{L+T}\int_T^{L+T} C_x(t_1-t_2)\, e^{-i\omega(t_1-t_2)}\, e^{-i(\omega_1+\omega_2)t_2}\, dt_1 dt_2$$

t_2



t_1

$\tau = t_1 - t_2$

$$= \int_{t_2=T}^{L+T} \int_{\tau} C_x(\tau)\, e^{-i\omega\tau}\, e^{-i(\omega_1+\omega_2)t_2}\, d\tau\, dt_2$$

Put $u = t_2 - T$.

$$= e^{-i(\omega_1+\omega_2)T} \int_{u=0}^{L} \int_{\tau} C_x(\tau)\, e^{-i\omega\tau}\, e^{-i(\omega_1+\omega_2)u}\, d\tau\, du.$$

This must (a) be indep of T + (b) be multiplied by $e^{-i(\omega_1+\omega_2)T}$ as T advances. So, $=0$ unless $\omega_1+\omega_2=0$

$$\therefore \langle F(x;\omega_1,L,T)\,F(x;\omega_2,L,T)\rangle = 0$$

unless $\omega_1+\omega_2=0$. if $\omega_1+\omega_2=0$,

$$F(x;\omega_2,L,T) = F(x;-\omega_1,L,T) = \overline{F(x;\omega_1,L,T)}$$

so the covariance is the power spectrum.

Bottom line: there's a natural description of the distribution of a time series, analogous to the variance of a scalar random variable

$$P_X(\omega) = \lim_{L\to\infty} \frac{1}{L}\left\langle \left| \int_0^L x(t)\,e^{-i\omega t}\,dt \right|^2 \right\rangle$$

* Ests of $P_X(\omega)$ are (approximately) independent across ω [depends on estimation procedure]
* $P_X(\omega)$ can be used to address questions on page ①.
* $P_X(\omega) = \int_{-\infty}^{\infty} C_X(\tau)\,e^{-i\omega\tau}\,d\tau$ [see pg ⑫ of "Power Spectra" Notes 2003-4]
* $P_X(\omega)$ does not fully characterize X

(variance does not fully characterize a distribution)

Brief discussion for further extensions of the power spectrum to characterize a noise:

A Consider $\langle F(x;\omega_1,L,T)\,F(x;\omega_2,L,T)\,F(x;\omega_3,L,T)\rangle$

analogous to ☆ on p. 9. Same argument shows that this $= 0$ unless

$$\omega_1+\omega_2+\omega_3=0.$$

So we can define the "bispectrum"

$$B_X(\omega_1,\omega_2) = \lim_{L\to\infty} \frac{1}{L^{3/2}} \left\langle F(x;\omega_1,L,T)\cdot F(x,\omega_2,L,T)\cdot \overline{F(x,\omega_1+\omega_2,L,T)} \right\rangle$$

Trispectrum, etc. can be similarly defined.

B Constructive proof that $P_X(\omega)$ does not characterize X fully:
(demo)

Given some $P(\omega)$, construct a noise X for which

$P_X(\omega)=P(\omega)$:

Choose a long L. Randomly choose $\hat{x}(\omega)$ (indep. for each ω)

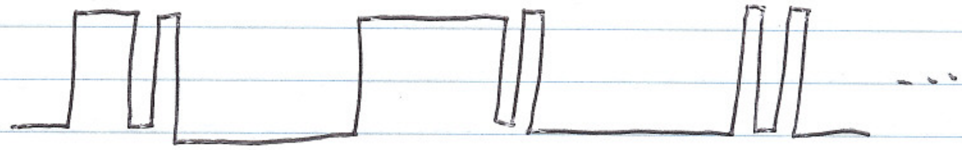
with $\langle|\hat{x}(\omega)|^2\rangle = L\,P(\omega)$. Then create $X(t)=\frac{1}{2\pi}\int e^{i\omega t}\hat{x}(\omega)\,d\omega$

If we choose $\hat{x}(\omega)$ from a Gaussian distribution in $\mathbb{C}$, then $x(t)$ will be Gaussian.

* need $\hat{x}(\omega)=\overline{\hat{x}(-\omega)}$ to guarantee $x(t)$ real.

(12)

But we could have started with a noise Y which was not Gaussian, e.g.

$y(t) =$ 

what we did was to find a Gaussian noise X for which

$$P_X(\omega) = P_Y(\omega).$$

Y differs from X in higher moments.

When signals are combined & filtered, what happens to their power spectra?

<u>Independent noises</u>: power spectra <u>add</u>.

$$P_Z(\omega) = P_X(\omega) + P_Y(\omega)$$

because

$$F(Z;\omega,L,T) = F(X;\omega,L,T) + F(Y;\omega,L,T)$$

so $$\langle F(Z)\overline{F(Z)}\rangle = \langle F(X)\overline{F(X)}\rangle + \langle F(X)\overline{F(Y)}\rangle + \langle F(Y)\overline{F(X)}\rangle + \langle F(Y)\overline{F(Y)}\rangle$$

Cross-terms $= 0$ because of independence.

NB: Say noises are not independent. Extreme case: $Y(t) = X(t)$. Then

$$P_Z(\omega) = 4P_X(\omega)$$

(13)

Filtering

$$x(t) \longrightarrow \boxed{L} \longrightarrow y(t)$$

$$P_y(\omega) = |L(\omega)|^2 P_x(\omega).$$

This is because each Fourier estimate for Y,

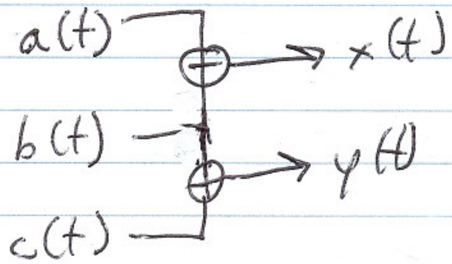
i.e., something of the form

$$\int_T^{L+T} y(t)\, e^{-i\omega t}\, dt ,$$

simply writes y in the Fourier basis, and $\tilde{y}(\omega) = \hat{L}(\omega)\tilde{X}(\omega)$

i.e the effect of L is to multiply a Fourier estimate of X by $\hat{L}(\omega)$

Common sources



a, b, c independent.

How to analyze this (& detect the common input?)

Time domain, – we'd look at $C_{XY}(\tau) = \langle X(t)Y(t+\tau)\rangle$.
("cross covariance")

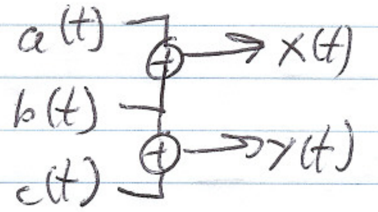
By analogy to univariate case, we will instead look at

$$P_{XY}(\omega) = \lim_{L\to\infty} \frac{1}{L}\left\langle F(x;\omega,L,T)\,\overline{F(y;\omega,L,T)}\right\rangle.$$

This is the "cross-spectrum". Same comments apply that one needn't consider $\langle F(x;\omega_1)\overline{F(y;\omega_2)}\rangle$, and that estimates are independent (approximately) at different ω.

[But $P_{XY}(\omega)$ need not be real.]

Back to



$P_X(\omega) = P_a(\omega) + P_b(\omega)$. $\quad P_Y(\omega) = P_b(\omega) + P_c(\omega)$.

$$P_{XY}(\omega) = \lim_{L\to\infty} \frac{1}{L}\left\langle \left(F(a;\omega,L,T) + F(b;\omega,L,T)\right)\cdot \overline{\left(F(b;\omega,L,T) + F(c;\omega,L,T)\right)}\right\rangle$$

$= P_b(\omega)$, just what we want.

Motivates def. of coherency $= \dfrac{P_{XY}(\omega)}{\sqrt{P_X(\omega)P_Y(\omega)}}$

& coherence $= |\text{coherency}|$.

Coherency + cross spectra need not be real.

Say $Y(t) = X(t - D)$, i.e., Y is a delayed version of X.

Then $$F(y; \omega, L, T) = \int_{T}^{T+L} y(t) e^{-i\omega t} dt$$

$$= \int_{T}^{T+L} x(t - D) e^{-i\omega t} dt$$

$$= \int_{T-D}^{T+L-D} x(u) e^{-i\omega(u+D)} du$$

$$= e^{-i\omega D} F(x; \omega, L, T - D)$$

so $\langle F(x; \omega, L, T) \overline{F(y; \omega, L, T)} \rangle$

$$= e^{i\omega D} \langle F(x; \omega, L, T) \overline{F(x, \omega, L, T - D)} \rangle$$

$$P_{XY}(\omega) = e^{i\omega D} P_X(\omega)$$

Coherency $= e^{i\omega D}$

Coherence $= 1$.