Noise and Variability

Homework #1 (2008), answers

Q1: Power spectra of some simple noises

A. Poisson noise. A Poisson noise $n(t)$ *is a sequence of delta-function pulses, each occurring independently, at some rate r. (More formally, it is a sum of pulses of width* $\Delta\tau$ *and height* $1/\Delta\tau$*, and the probability of a pulse between time t and* $t+\Delta t$ *is* $r\Delta t$*, and we consider the limit of* $\Delta\tau \to 0$ *and* $\Delta t \to 0$*). Calculate the power spectrum* $P_n(\omega)$ *of this noise.*

Solution.
The straightforward approach is surprisingly difficult because one has to be careful about the limits.

Start with the definition $P_n(\omega) = \lim_{L\to\infty} \frac{1}{L}\left\langle \left| \int_0^L n(t)e^{-i\omega t}dt \right|^2 \right\rangle$. We consider first $\omega \neq 0$; for $\omega = 0$ we have to remember the "fine print" that we need to subtract a constant value (here, r) from $n(t)$ so that its mean is 0.

Take a small Δt, and approximate the integral by a sum over a large number $N = L/\Delta t$ of intervals of length Δt, i.e., $\int_0^L n(t)e^{-i\omega t}dt \approx \sum_{k=1}^{L/\Delta t} n_k e^{-i\omega k\Delta t}$, where $n_k = 1$ if there is an even in the kth interval, and $n_k = 0$ if there is no event. Each n_k is independent, and the probability of $n_k = 1$ is $r\Delta t$.

$$\frac{1}{L}\left|\int_0^L n(t)e^{-i\omega t}dt\right|^2 \approx \frac{1}{L}\left|\sum_{k=1}^{N} n_k e^{-i\omega k\Delta t}\right|^2 = \frac{1}{L}\left(\sum_{k=1}^{N} n_k e^{-i\omega k\Delta t}\right)\left(\sum_{k=1}^{N} n_k e^{i\omega k\Delta t}\right) = \frac{1}{L}\left(\sum_{k,m=1}^{N} n_k n_m e^{-i\omega(k-m)\Delta t}\right).$$

The last quantity is best analyzed according to whether $m = k$ or not. For $m = k$, $n_k n_m = n_k^2 = n_k$ (since n_k can only be 0 or 1), so the expected value of each term is $r\Delta t$. There are $N = L/\Delta t$ values of k, so their contribution to $P_n(\omega)$ is $\frac{1}{L}N(r\Delta t) = r$.

The harder part is to show, rigorously, that the contribution of the terms in $\frac{1}{L}\left(\sum_{k,m=1}^{N} n_k n_m e^{-i\omega(k-m)\Delta t}\right)$ for which $m \neq k$ do not contribute. Say $n = m-k$. There are $N - |n|$ pairs of m and k in the sum, for which $n = m-k$. Since n_k and n_m are independent and each has a probability of $r\Delta t$ of being equal to 1, then their product has a probability of $(r\Delta t)^2$ of begin equal to 1. So the portion of the sum for which $n = m-k$ has an expected value of

$\left\langle \frac{1}{L}\left(\sum_{k-m=n} n_k n_m e^{-i\omega(k-m)\Delta t} \right) \right\rangle = \frac{1}{L}(N-|n|)(r\Delta t)^2 e^{-i\omega n\Delta t} = r^2\Delta t(1-\frac{|n|}{N})e^{-i\omega n\Delta t}$. Summing this over the entire range of n (from $-N$ to N) and letting $\Delta t \to 0$ (with $N = L/\Delta t$ and $n = t/\Delta t$) yields

$\lim_{\Delta t\to 0}\sum_{n=-N}^{N} r^2\Delta t(1-\frac{|n|}{N})e^{-i\omega n\Delta t} = r^2\int_{-L}^{L}(1-\frac{|t|}{L})e^{-i\omega t}dt$. This last integral is the Fourier transform of the triangle-wave (see Linear Systems Theory homework Q2), so

$r^2\int_{-L}^{L}(1-\frac{|t|}{L})e^{-i\omega t}dt = r^2 L\int_{-L}^{L} f_{triangle}(t)e^{-i\omega t}dt = r^2 L\tilde{f}_{triangle}(\omega) = r^2 L\left(\frac{\sin(\omega L/2)}{\omega L/2}\right)^2$. Provided $\omega \neq 0$, this goes to 0 (as $1/L$) for sufficiently large L.

For $\omega = 0$, the above expression does not go to 0; in fact, it diverges (as L). The problem is that at $\omega = 0$, it matters that the mean value of the signal $n(t)$ was nonzero – it is r. We can only expect a convergent value for $P_n(\omega) = \lim_{L\to\infty}\frac{1}{L}\left\langle \left| \int_0^L n(t)e^{-i\omega t}dt \right|^2 \right\rangle$ if we start with a signal of mean 0. The $\omega = 0$-case for $P_n(\omega) = \lim_{L\to\infty}\frac{1}{L}\left\langle \left| \int_0^L \left(n(t) - < n(t) > \right)e^{-i\omega t}dt \right|^2 \right\rangle$ is best handled by a special argument.

We want to find $P_n(0) = \lim_{L\to\infty}\frac{1}{L}\left\langle \left| \int_0^L \left(n(t)-r\right)dt \right|^2 \right\rangle$.

$\int_0^L \left(n(t)-r\right)dt = \int_0^L n(t)dt - rL$. The first term counts the number of Poisson events in a segment of length L; rL is the expected mean of this quantity.

So $\left\langle \left| \int_0^L \left(n(t)-r\right)dt \right|^2 \right\rangle$ is the variance of a Poisson distribution with mean rL, namely, rL

(The variance of a Poisson distribution is equal to its mean.) So $P_n(0) = \frac{rL}{L} = r$.

Summarizing: For $\omega \neq 0$, the self-terms of $\frac{1}{L}\left(\sum_{k,m=1}^{N} n_k n_m e^{-i\omega(k-m)\Delta t} \right)$ contribute r to $P_n(\omega)$, and the cross-terms have a limit of 0. For $\omega = 0$, an argument based on Poisson counting statistics shows that $P_n(0) = r$. So for all ω, $P_n(\omega) = r$.

Alternative partial solution with a useful insight.

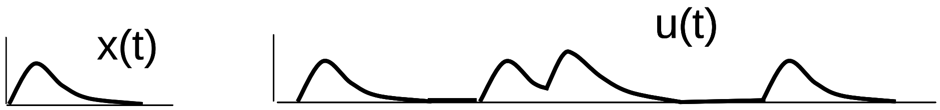
Recognize that (i) a faster Poisson process is also a Poisson process, and (ii) speeding up the process is a change of scale of the power spectrum. Therefore,

$$P_n(c\omega) = \lim_{L\to\infty}\frac{1}{L}\left\langle\left|\int_0^L n(t)e^{-ic\omega t}dt\right|^2\right\rangle = \lim_{L\to\infty}\frac{1}{L}\left\langle\left|\frac{1}{c}\int_0^{cL} n(\frac{u}{c})e^{-i\omega u}du\right|^2\right\rangle = \lim_{L\to\infty}\frac{1}{c^2L}\left\langle\left|\int_0^{cL} n(\frac{u}{c})e^{-i\omega u}du\right|^2\right\rangle$$

where we've used $u = ct$, and

$$\lim_{L\to\infty}\frac{1}{c^2L}\left\langle\left|\int_0^{cL} n(\frac{u}{c})e^{-i\omega u}du\right|^2\right\rangle = \frac{1}{c}\lim_{M\to\infty}\frac{1}{M}\left\langle\left|\int_0^{M} n(\frac{u}{c})e^{-i\omega u}du\right|^2\right\rangle$$ where we've used $M = cL$. The Poisson process represented by $n(u/c)$ has a rate *rc* (with time measured by *u*), and thus can be regarded as a sum of *c* independent Poisson processes of rate *r*. So the power spectrum of the Poisson process $n(u/c)$ is *c* times larger than the power spectrum *n*(*t*). So $P_n(c\omega) = \frac{1}{c}P_{n(u/c)}(\omega) = \frac{c}{c}P_n(\omega)$, i.e., $P_n(\omega)$ is independent of ω. But it still takes some work to find this value. For example, let $\omega \to \infty$. Then each nonzero term in the Fourier estimate is dephased, and has magnitude 1. This shows that as $\omega \to \infty$, $P_n(\omega)$ is the average number of events per unit time, i.e., *r*.

B. Shot noise. A shot noise $u(t)$ *is a process in which copies of a stereotyped waveform* $x(t)$*, occurring at random times, are superimposed. That is,* $u(t) = \sum_{t_i} x(t - t_i)$*, where the times* t_i *are determined by a Poisson process of rate* r*. The "shots"* $x(t)$ *are typically considered to be causal, namely,* $x(t) = 0$ *for* $t < 0$*. Given the Fourier transform* $\tilde{u}(\omega) = \int_0^\infty u(t)e^{-i\omega t}dt$*, find the power spectrum* $P_u(\omega)$ *of u.*



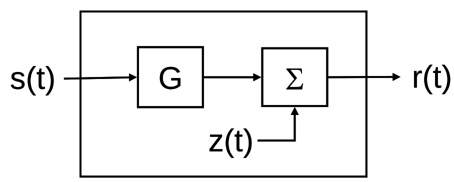
Solution.
The shot noise process is the result of filtering a Poisson process (of rate *r*) by the linear filter with impulse response $x(t)$. By part A, the Poisson process has power spectrum *r*. Filtering a signal by a linear filter with transfer function $\tilde{X}(\omega)$ multiplies the power spectrum by $\left|\tilde{X}(\omega)\right|^2$. So $P_u(\omega) = r\left|\tilde{X}(\omega)\right|^2$.

C. Shot noise, variable shot size. This is a process $v(t)$ *in which the amplitudes of the "shots" vary randomly. That is,* $v(t) = \sum_{t_i} a_i x(t - t_i)$*, where the amplitudes* a_i *are chosen* *independently. Given the Fourier transform* $\tilde{u}(\omega)=\int_0^\infty u(t)e^{-i\omega t}dt$ *and the moments of the distribution of the* a_i*, find the power spectrum* $P_v(\omega)$ *of v.*

Solution. We can use the reasoning of part B, but applied to a modified Poisson process that has impulses of amplitudes a_i. We calculate the power spectrum of this process by the method of part A. Since these amplitudes are independent, every step of part A is readily extended, yielding the result that the power spectrum of the modified Poisson process is $r\langle a^2\rangle$. So $P_v(\omega)=r\langle a^2\rangle\left|\tilde{X}(\omega)\right|^2$.

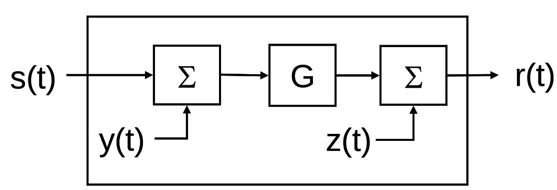
Q2: Input and output noise

Recall the behavior of a linear system with additive noise (pages 16-17 of NAV notes), consisting of a linear filter G (characterized by its transfer function $\tilde{g}(\omega)$*:*

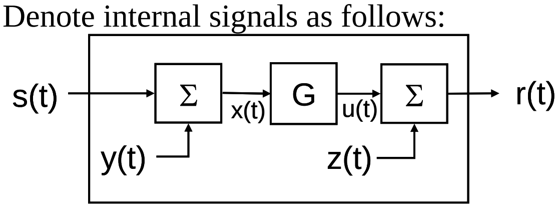


If the input is $s(t)=\tilde{s}(\omega_0)e^{i\omega_0 t}$ *and there is an additive noise* $z(t)$ *with power spectrum* $P_z(\omega)$*, then the quantity* $\frac{1}{T}F(r,\omega_0,T,0)\equiv\frac{1}{T}\int_0^T r(t)e^{-i\omega_0 t}dt$*, when calculated for data lengths T that are a multiple of the period* $2\pi/\omega_0$*, has a mean value* $\tilde{s}(\omega_0)\tilde{g}(\omega_0)$ *and a variance* $\frac{1}{T}P_z(\omega_0)$.

Analyze the situation when there is also some noise added prior to G, diagrammed below:



Solution:
Denote internal signals as follows:



Based on the simpler system considered on pages 16-17, signals $x(t)$ are characterized by $\frac{1}{T}F(x,\omega_0,T,0) \equiv \frac{1}{T}\int_0^T x(t)e^{-i\omega_0 t}dt$, which has mean $\tilde{s}(\omega_0)$ and variance $\frac{1}{T}P_y(\omega_0)$. (The system consisting of *s*, *y*, and *x* is identical to the one on page 16-17, but with $G = I$.)
Fourier components of the signal at *u* are equal to those at *x*, multiplied by $\tilde{g}(\omega)$.

Therefore, signals at u are characterized by $\frac{1}{T}F(u,\omega_0,T,0) \equiv \frac{1}{T}\int_0^T u(t)e^{-i\omega_0 t}dt$ with a mean $\tilde{s}(\omega_0)\tilde{g}(\omega_0)$ and variance $\frac{1}{T}P_y(\omega_0)\left|g(\omega_0)\right|^2$.
Adding an independent noise term $z(t)$ does not change the mean, but adds to the variance according to its power spectrum. So $\frac{1}{T}F(r,\omega_0,T,0) \equiv \frac{1}{T}\int_0^T r(t)e^{-i\omega_0 t}dt$ has mean $\tilde{s}(\omega_0)\tilde{g}(\omega_0)$ and variance $\frac{1}{T}\left(P_y(\omega_0)\left|g(\omega_0)\right|^2 + P_z(\omega_0)\right)$.