26

Fourier Analysis - Application Noise & Variability III

Channel Noise

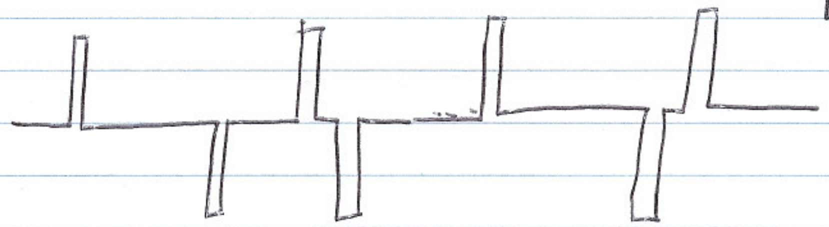
Simplest case: $n(t) =$ 

Probability of switching per unit time $= r$.

Calculate the power spectrum.

Instead, calculate p.s. of $\frac{dn}{dt} = q(t)$; then $P_n(\omega) = \left|\frac{1}{i\omega}\right|^2 P_q(\omega) = \frac{P_q(\omega)}{\omega^2}$,

since $\hat{n}(\omega) = \frac{1}{i\omega}\hat{q}(\omega)$. [LST HW 1C]

$q(t) =$ 

a random sequence of δ-functions, but alternating in sign.

As in NAV homework Q1, the power spectrum of q is given by

$$P_q(\omega) = r + \lim_{\Delta t \to 0} \sum_{n=-N}^{N} r^2 \Delta t \left(1 - \frac{|n|}{N}\right) e^{-i\omega n \Delta t}\, c(n\Delta t)$$

(the $\Delta t = 0$ term)

where $c(n\Delta t)$ is the correlation of events separated by $n\Delta t$, each event considered as 1, ~~or~~ -1.

$$P_q(\omega) = r + \lim_{L\to\infty} r^2 \int_{-L}^{L} \left(1 - \frac{|t|}{L}\right) c(t)\, e^{-i\omega t}\, dt$$

$L = N\Delta t$, $t = n\Delta t$.

(27)

$c(0) = 1$ (doesn't influence the integral)

$t>0$: $c(t) = -1\cdot$ probability of no events in $(0,t)$

$+1\cdot$ " " 1 " " $(0,t)$

$-1\cdot$ " " 2 " " $(0,t)$

etc.

$$c(t) = -1\cdot e^{-rt} + (rt)e^{-rt} - \frac{(rt)^2}{2} e^{-rt}$$

$$= e^{-rt}\left(-1 + rt - \frac{(rt)^2}{2!} + \frac{(rt)^3}{3!} \dots\right)$$

$$= e^{-rt}\cdot(-e^{-rt}) = e^{-2rt}.$$

$t<0$: $c(t) = c(|t|) = e^{-2r|t|}$

$$P_q(\omega) = r + \lim_{L\to\infty} -r^2 \int_{-L}^{L} \left(1 - \frac{|t|}{L}\right) e^{-i\omega t} e^{-2r|t|} dt$$

$e^{-2r|t|}$ is only large if $|t| << \frac{1}{r}$. So $\frac{|t|}{L} << \frac{1}{rL}$.

So that term can be neglected as $L\to\infty$.

$$P_q(\omega) = r - r^2 \int_{-\infty}^{\infty} e^{-i\omega t} e^{-2r|t|} dt$$

$$= r - r^2 \cdot 2\,\mathrm{Re} \int_0^{\infty} e^{-i\omega t - 2rt} dt \quad \downarrow \text{use symmetry of } \int$$

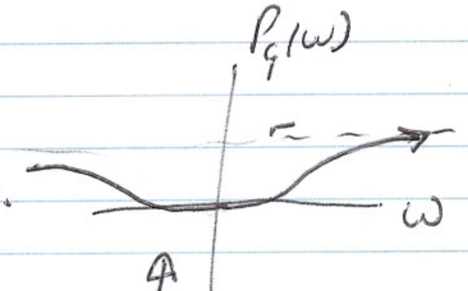
$$= r - r^2\cdot 2\,\mathrm{Re} \left.\frac{e^{-i\omega t - 2rt}}{-i\omega - 2r}\right|_0^{\infty}$$

(28)

$$P_q(\omega) = r - r^2 \cdot 2\,\mathrm{Re}\,\frac{1}{i\omega+2r}$$

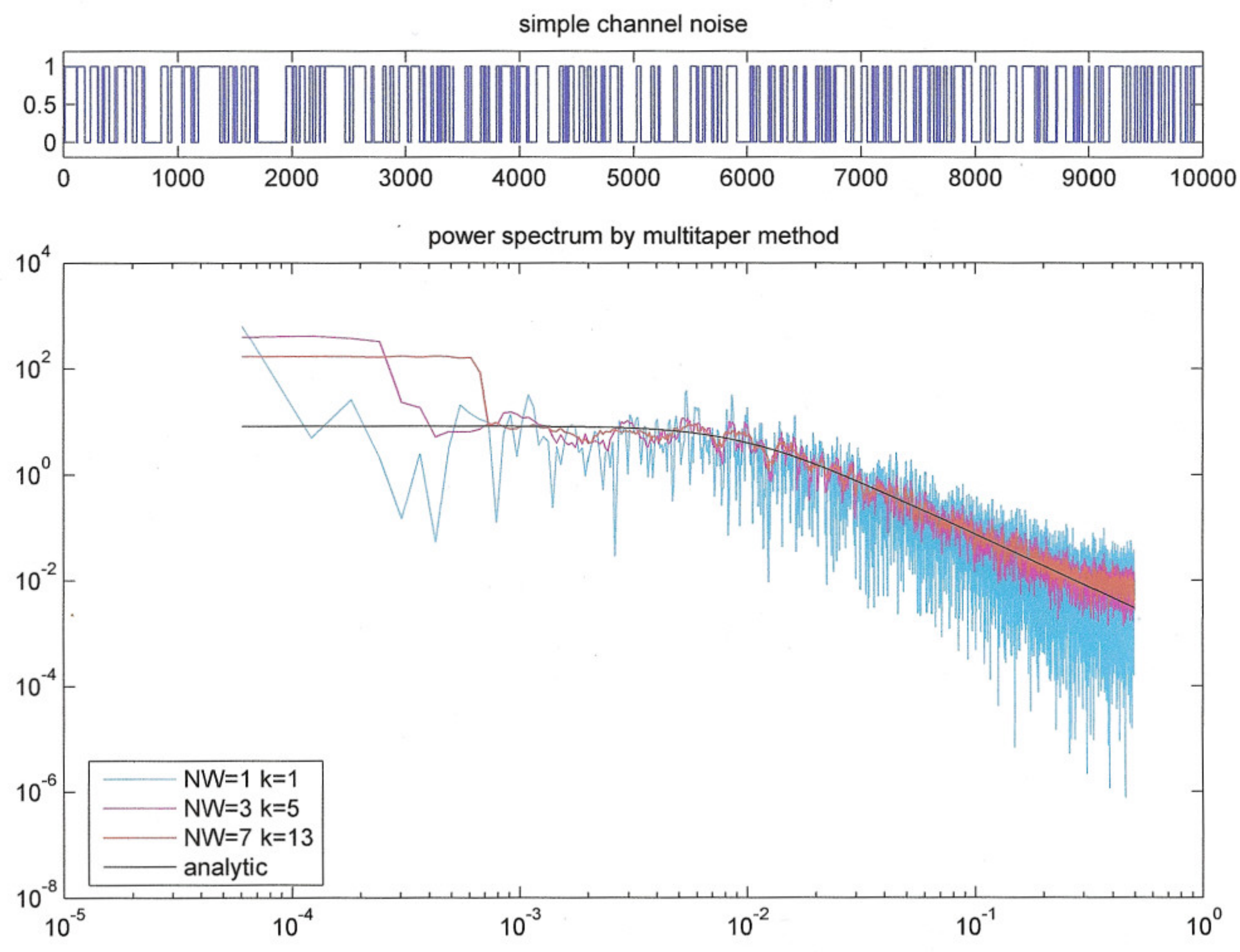
$$= r - r^2\left(\frac{1}{i\omega+2r}+\frac{1}{-i\omega+2r}\right)$$

$$= r - r^2\left(\frac{4r}{\omega^2+4r^2}\right) = \frac{r\omega^2}{4r^2+\omega^2}.$$



The low-freq dip reflects the "balance" of $q(\omega)$:

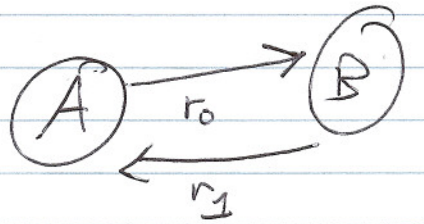
$$\boxed{P_n(\omega) = \frac{1}{\omega^2}\,P_q(\omega) = \frac{r}{4r^2+\omega^2}.}$$



24

Method 2.

State machine.



$A =$ "closed", produces a value of 0

$B =$ "open", produces a value of 1. Note r_0 need not $= r_1$.

This generates a time series $n(t)$. Find $P_n(\omega)$.

We will do this by finding the autocorrelation

$$c_n(\tau) = \langle n(t)\, n(t+\tau) \rangle \quad \text{and then use}$$

$$P_n(\omega) = \int_{-\infty}^{\infty} c_n(\tau)\, e^{-i\omega\tau}\, d\tau .$$

$$c_n(\tau) = \langle n(t) n(t+\tau) \rangle - \langle n(t) \rangle^2 .$$

Say $p(A/A)(\tau) =$ probability that system is in state A at time τ, given that it was initialized in state A at time 0.

$p(B/A)(\tau) =$ prob. that system is in state B at time τ, given that it was initialized in state A at time 0.

$p(A/B)$, $p(B/B)$ etc. Must have
$$\begin{cases} p(A/A)(\infty) = p(A/B)(\infty) = p(0) \\ p(B/A)(\infty) = p(B/B)(\infty) = p(1) \\ p(A/A) + p(B/A) = 1 \\ p(A/B) + p(B/B) = 1 \end{cases}$$

$$c_n(\tau) = p(B/B)(\tau) - p(B/B)^2(\infty).$$

$c_n(\infty) = 0.$ $\quad c_n(0) = (1 - \text{mean})^2 \cdot p(1) + (0 - \text{mean})^2 \cdot p(0)$

$= p(1) \cdot p(0)$ $\quad$ [mean $= p(1)$]

(30)

Write differential eqlns for $p(A|A)$, etc:

$$\frac{d}{d\tau} p(A|X)(\tau) = -r_0\, p(A|X)(\tau) + r_1\, p(B|X)(\tau)$$

$$\frac{d}{d\tau} p(B|X)(\tau) = r_0\, p(A|X)(\tau) - r_1\, p(B|X)(\tau).$$

steady-state: $p(A|X)(\tau) = \frac{r_1}{r_0}\, p(B|X)(\tau)$

$$p(A|X)(\tau) + p(B|X)(\tau) = 1$$

so $p(A|X) = \frac{r_1}{r_0+r_1}$, $p(B|X) = \frac{r_0}{r_1+r_0}$.

subtracting

$$\frac{d}{d\tau}\left(p(A|X) - p(B|X)\right) = -(r_0+r_1)\left(p(A|X) - p(B|X)\right)$$

so $p(A|X) - p(B|X)$ evolves like $e^{-(r_0+r_1)t}$.

so $C_n(\tau) = K e^{-(r_0+r_1)|t|}$, $K = \frac{r_0 r_1}{(r_0+r_1)^2}$.

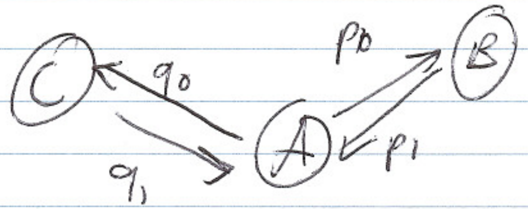
$$\text{so } P_n(\omega) = \int_{-\infty}^{\infty} C_n(\omega) e^{-i\omega\tau}\, d\tau = \frac{r_0 r_1}{(r_0+r_1)^2} \int_{-\infty}^{\infty} e^{-(r_0+r_1)|t|} e^{-i\omega\tau}\, d\tau$$

$$= \frac{r_0 r_1}{(r_0+r_1)^2}\left[\frac{1}{(r_0+r_1)+i\omega} + \frac{1}{(r_0+r_1)-i\omega}\right] = \frac{r_0 r_1}{(r_0+r_1)^2}\left(\frac{2(r_0+r_1)}{(r_0+r_1)^2+\omega^2}\right)$$

$$= \frac{r_0 r_1}{(r_0+r_1)/2}\, \frac{1}{(r_0+r_1)^2+\omega^2}.$$

"Method 3."

Multiple States.



Say each state (H = A, B, C, ...) leads to a signal 1 size V_H.

$$\langle n \rangle = \sum_H V_H \cdot p_\infty(H)$$

$$G_n(\tau) = \langle n(t)\, n(t+\tau) \rangle = \sum_{H,K} V_H V_K \, \text{prob}\left(\text{state } K \text{ at time } \tau \,\middle|\, \text{state } H \text{ at time } 0\right) \cdot p_\infty(H)$$

$$\text{prob}\begin{pmatrix} A \text{ at time } \tau+\Delta\tau \mid H \\ B \text{ at time } \tau+\Delta\tau \mid H \\ \vdots \\ C \text{ at time } \tau+\Delta\tau \mid H \end{pmatrix} = \begin{pmatrix} 1-p_0\Delta\tau-q_0\Delta\tau & p_1\Delta\tau & q_1\Delta\tau \\ p_0\Delta\tau & 1-p_1\Delta\tau & 0 \\ q_0\Delta\tau & 0 & 1-q_1\Delta\tau \end{pmatrix} \cdot \text{prob}\begin{pmatrix} A \text{ at time } \tau \mid H \\ B \ldots \tau \mid H \\ C \text{ at time } \tau \mid H \end{pmatrix}$$

$$X(\tau+\Delta\tau) = (I + \Delta\tau \cdot E)\, X(\tau); \quad E = \begin{pmatrix} -p_0-q_0 & p_1 & q_1 \\ p_0 & -p_1 & 0 \\ q_0 & 0 & -q_1 \end{pmatrix}$$

$$X(\tau+\Delta\tau) - X(\tau) = \Delta\tau \cdot E\, X(\tau)$$

$$\frac{dX}{d\tau} = E\, X(\tau)$$

We'd like a solution of $\frac{dx(\tau)}{d\tau} = E\,x(\tau)$.

Formal solution: $x(\tau) = e^{Et}\,x(0)$.

$$\Big[\text{or, } x(\tau+\Delta\tau) = \lim_{N\to\infty}\left(I + \frac{\Delta\tau}{N}E\right)^N x(\tau)$$

$$= e^{\Delta\tau E}\,x(\tau)\Big]$$

But how to compute $e^{Et}x(0)$?

Say E has an eigenvector v, with eigenvalue λ.

$$e^{Et}v = \left(I + Et + \frac{E^2t^2}{2!} + \frac{E^3t^3}{3!} + \cdots\right)v$$

$$= Iv + tEv + \frac{t^2}{2!}E^2v + \frac{t^3}{3!}E^3v + \cdots$$

$$= v + t\lambda v + \frac{t^2}{2!}\lambda^2 v + \frac{t^3}{3!}\lambda^3 v + \cdots$$

$$= \left(1 + t\lambda + \frac{t^2}{2!}\lambda^2 + \frac{t^3}{3!}\lambda^3 + \cdots\right)v = e^{t\lambda}v.$$

So, if we have all the eigenvectors of E, and they form a basis, we can write $x(0) = \sum \alpha_i v_i$, and

$$e^{Et}x(0) = e^{Et}\sum \alpha_i v_i = \sum \alpha_i e^{Et} v_i = \sum \alpha_i e^{t\lambda_i} v_i$$

$C_n(\tau)$ is ∴ a sum of exponentials $\sum \beta_i e^{-|t|\lambda_i}$

NB: One $\lambda = 0$, since $E\begin{pmatrix}1\\1\\1\end{pmatrix} = 0$.

All other λ_i have real part < 0.

$$\therefore \ P_n(\omega) = \sum_i \beta_i \frac{2\lambda_i}{\lambda_i^2 + \omega^2}.$$