Nonlinear Systems Theory

Homework #1 (2008) Answers

Laguerre Polynomials. These are classically defined as the orthonormal polynomials with respect to the weight $\exp(-x)$ for $x \geq 0$. Here we calculate orthogonal (not necessarily orthonormal) polynomials with respect to a scaled version of that weight, namely, $W(x) = \frac{1}{b}\exp(-x/b)$ for $x \geq 0$.

Q1: Find an expression for the moments, $M_n = \int_0^\infty x^m \left(\frac{1}{b}e^{-x/b}\right)dx$, for $n \leq 5$ (or, in general).

Answers:

First, note that $M_n = \int_0^\infty x^n \left(\frac{1}{b}e^{-x/b}\right)dx = b^n \int_0^\infty y^n e^{-y}dy$, after the substitution $y = x/b$, so $M_n = b^n I_n$ where $I_n = \int_0^\infty y^n e^{-y}dy$. So we need to find I_n.

Method 1. The gamma function. The gamma-function is a classical special function defined by $\Gamma(z) = \int_0^\infty y^{z-1}e^{-y}dy$. So $I_n = \Gamma(n+1)$. An elementary property of the gamma-function is that $\Gamma(n+1) = n!$ for non-negative integers *n*. But this is really a cheap appeal to authority, since the properties of the gamma function are proven by along the lines of the methods spelled out below.

Method 2. Integration by parts. Note that $I_0 = \int_0^\infty e^{-y}dy = -e^{-y}\Big|_0^\infty = 1$.

Then, taking $u = y^n$, $dv = e^{-y}dy$, so $du = ny^{n-1}dy$, $v = -e^{-y}$, and the integration-by-parts formula $\int udv = uv - \int vdu$ yields (for $n \geq 1$)

$I_n = \int_0^\infty y^n e^{-y}dy = -y^n e^{-y}\Big|_0^\infty + n\int_0^\infty y^{n-1}e^{-y}dy = nI_{n-1}$. So $I_n = n!$.

Method 3. Generating functions. Let $J(z) = \sum_{n=0}^\infty \frac{z^n}{n!}I_n$. Then

$J(z) = \sum_{n=0}^\infty \frac{z^n}{n!}\int_0^\infty y^n e^{-y}dy = \sum_{n=0}^\infty \int_0^\infty \frac{z^n y^n}{n!}e^{-y}dy = \int_0^\infty e^{yz}e^{-y}dy = \int_0^\infty e^{-y(1-z)}dy = \frac{1}{1-z}$. Matching coefficients in $J(z) = \sum_{n=0}^\infty \frac{z^n}{n!}I_n$ and $J(z) = \frac{1}{1-z} = \sum_0^\infty z^n$ again yields $I_n = n!$

Q2: Carry out the Gram-Schmidt procedure, for the polynomials x^0, x^1, x^2, x^3, with an inner product defined by the Laguerre weight, $\langle f,g\rangle = \int_0^\infty f(x)g(x)\left(\frac{1}{b}e^{-x/b}\right)dx$.

Answer:

$\varphi_0 = x^0 = 1$;

for the next step: $\langle \varphi_0, \varphi_0\rangle = M_0 = 1$.

$$\varphi_1 = x^1 - \frac{\langle x^1, \varphi_0\rangle}{\langle \varphi_0, \varphi_0\rangle}\varphi_0 = x^1 - \frac{M_1}{M_0}x^0 = x - b\ ;$$

for the next step: $\langle \varphi_1, \varphi_1\rangle = M_2 - 2bM_1 + b^2 = b^2(2! - 2 + 1) = b^2$.

$$\varphi_2 = x^2 - \frac{\langle x^2, \varphi_1\rangle}{\langle \varphi_1, \varphi_1\rangle}\varphi_1 - \frac{\langle x^2, \varphi_0\rangle}{\langle \varphi_0, \varphi_0\rangle}\varphi_0 = x^2 - \frac{M_3 - M_2 b}{b^2}(x-b) - \frac{M_2}{1}1$$

$$\varphi_2 = x^2 - \frac{6b^3 - 2b^3}{b^2}(x-b) - 2b^2 = x^2 - 4b(x-b) - 2b^2 = x^2 - 4bx + 2b^2 .$$

for the next step:

$$\langle \varphi_2, \varphi_2\rangle = \langle \varphi_2, x^2\rangle = M_4 - 4bM_3 + 2b^2M_2 = b^4(4! - 4\times 3! + 2\times 2!) = b^4(24 - 24 + 4) = 4b^4 .$$

(The first equality follows since the difference between φ_2 and x^2 is orthogonal to φ_2.)

$$\varphi_3 = x^3 - \frac{\langle x^3, \varphi_2\rangle}{\langle \varphi_2, \varphi_2\rangle}\varphi_2 - \frac{\langle x^3, \varphi_1\rangle}{\langle \varphi_1, \varphi_1\rangle}\varphi_1 - \frac{\langle x^3, \varphi_0\rangle}{\langle \varphi_0, \varphi_0\rangle}\varphi_0$$

$$\varphi_3 = x^3 - \frac{M_5 - 4bM_4 + 2b^2M_3}{4b^4}(x^2 - 4bx + 2b^2) - \frac{M_4 - bM_3}{b^2}(x-b) - \frac{M_3}{1}1$$

$$\varphi_3 = x^3 - b\frac{5! - 4\times 4! + 2\times 3!}{4}(x^2 - 4bx + 2b^2) - b^2\frac{4! - 3!}{1}(x-b) - 6b^3$$

$$\varphi_3 = x^3 - 9b(x^2 - 4bx + 2b^2) - 18b^2(x-b) - 6b^3 = x^3 - 9bx^2 + 18b^2x - 6b^3$$

for the next step:

$$\langle \varphi_3, \varphi_3\rangle = \langle \varphi_3, x^3\rangle = M_6 - 9bM_5 + 18b^2M_4 - 6b^3M_3 = b^6(6! - 9\times 5! + 18\times 4! - 6\times 3!) = .$$

$$= b^6(720 - 1080 + 432 - 36) = 36b^6$$