Nonlinear Systems Theory

Homework #1 (2008) Answers

Laguerre Polynomials. These are classically defined as the orthonormal polynomials with respect to the weight \( \exp(-x) \) for \( x \geq 0 \). Here we calculate orthogonal (not necessarily orthonormal) polynomials with respect to a scaled version of that weight, namely, \( W(x) = \frac{1}{b} \exp(-x/b) \) for \( x \geq 0 \).

Q1: Find an expression for the moments, \( M_n = \int_0^\infty x^n \left( \frac{1}{b} e^{-x/b} \right) dx \), for \( n \leq 5 \) (or, in general).

Answers:

First, note that \( M_n = \int_0^\infty x^n \left( \frac{1}{b} e^{-x/b} \right) dx = b^n \int_0^\infty y^n e^{-y} dy \), after the substitution \( y = x/b \), so 
\[ M_n = b^n I_n \]
where \( I_n = \int_0^\infty y^n e^{-y} dy \). So we need to find \( I_n \).

Method 1. The gamma function. The gamma-function is a classical special function defined by \( \Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy \). So \( I_n = \Gamma(n+1) \). An elementary property of the gamma-function is that \( \Gamma(n+1) = n! \) for non-negative integers \( n \). But this is really a cheap appeal to authority, since the properties of the gamma function are proven by along the lines of the methods spelled out below.

Method 2. Integration by parts. Note that \( I_0 = \int_0^\infty e^{-y} dy = \left[ -e^{-y} \right]_0^\infty = 1 \).

Then, taking \( u = y^n \), \( dv = e^{-y} dy \), so \( du = ny^{n-1} dy \), \( v = -e^{-y} \), and the integration-by-parts formula \( \int udv = uv - \int vdu \) yields (for \( n \geq 1 \))
\[ I_n = \int_0^\infty y^n e^{-y} dy = -y^n e^{-y} \bigg|_0^n + n \int_0^\infty y^{n-1} e^{-y} dy = nI_{n-1} \]. So \( I_n = n! \).

Method 3. Generating functions. Let \( J(z) = \sum_{n=0}^\infty \frac{z^n}{n!} I_n \). Then
\[ J(z) = \sum_{n=0}^\infty \frac{z^n}{n!} \int_0^\infty y^n e^{-y} dy = \sum_{n=0}^\infty \frac{z^n}{n!} e^{-y} \bigg|_0^\infty = \sum_{n=0}^\infty e^{z^n} e^{-y} dy = \int_0^\infty e^{-y(1-z)} dy = \frac{1}{1-z} \].

Matching coefficients in \( J(z) = \sum_{n=0}^\infty \frac{z^n}{n!} I_n \) and \( J(z) = \frac{1}{1-z} = \sum_{n=0}^\infty z^n \) again yields \( I_n = n! \).
Q2: Carry out the Gram-Schmidt procedure, for the polynomials \(x^0, x^1, x^2, x^3\), with an inner product defined by the Laguerre weight, \(\langle f, g \rangle = \int_0^\infty f(x)g(x)\left(\frac{1}{b} e^{-x/b}\right)dx\).

Answer:
\(\varphi_0 = x^0 = 1\);
for the next step: \(\langle \varphi_0, \varphi_0 \rangle = M_0 = 1\).

\[\varphi_1 = x^1 - \left(\frac{\langle x^1, \varphi_0 \rangle}{\langle \varphi_0, \varphi_0 \rangle}\right) \varphi_0 = x^1 - \frac{M_1}{M_0} x^0 = x - b;\]
for the next step: \(\langle \varphi_1, \varphi_1 \rangle = M_2 - 2bM_1 + b^2 = b^2(2! - 2 + 1) = b^2.\)

\[\varphi_2 = x^2 - \left(\frac{\langle x^2, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle}\right) \varphi_1 - \left(\frac{\langle x^2, \varphi_0 \rangle}{\langle \varphi_0, \varphi_0 \rangle}\right) \varphi_0 = x^2 - \frac{M_3 - M_2 b}{b^2} (x - b) - \frac{M_2}{1};\]
\[\varphi_2 = x^2 - \frac{6b^3 - 2b^3}{b^2} (x - b) - 2b^2 = x^2 - 4b(x - b) - 2b^2 = x^2 - 4bx + 2b^2.\]
for the next step:
\(\langle \varphi_2, \varphi_2 \rangle = \langle \varphi_2, x^2 \rangle = M_4 - 4bM_3 + 2b^2 M_2 = b^4(4! - 4 \times 3! + 2 \times 2!) = b^4(24 - 24 + 4) = 4b^4.\)
(The first equality follows since the difference between \(\varphi_2\) and \(x^2\) is orthogonal to \(\varphi_2\).)

\[\varphi_3 = x^3 - \left(\frac{\langle x^3, \varphi_2 \rangle}{\langle \varphi_2, \varphi_2 \rangle}\right) \varphi_2 - \left(\frac{\langle x^3, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle}\right) \varphi_1 - \left(\frac{\langle x^3, \varphi_0 \rangle}{\langle \varphi_0, \varphi_0 \rangle}\right) \varphi_0 = x^3 - \frac{M_5 - 4bM_4 + 2b^2 M_3}{4b^4} (x^2 - 4bx + 2b^2) - \frac{M_4 - bM_3}{b^2} (x - b) - \frac{M_3}{1};\]
\[\varphi_3 = x^3 - b \left(\frac{5! - 4 \times 4! + 2 \times 3!}{4} \right) (x^2 - 4bx + 2b^2) - b^2 \left(\frac{4! - 3!}{1}\right) (x - b) - 6b^3;\]
\[\varphi_3 = x^3 - 9b(x^2 - 4bx + 2b^2) - 18b^2(x - b) - 6b^3 = x^3 - 9bx^2 + 18b^2x - 6b^3.\]
for the next step:
\(\langle \varphi_3, \varphi_3 \rangle = \langle \varphi_3, x^3 \rangle = M_6 - 9bM_5 + 18b^2M_4 - 6b^3 M_3 = b^6(6! - 9 \times 5! + 18 \times 4! - 6 \times 3!) = b^6(720 - 1080 + 432 - 36) = 36b^6.\)