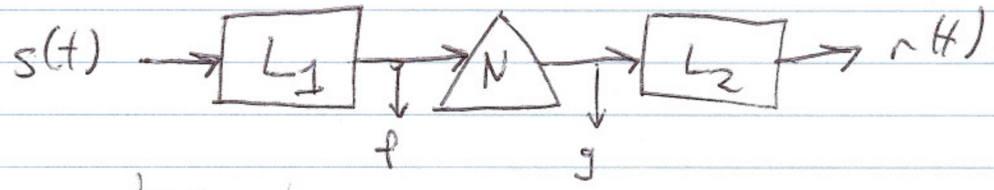
Nonlinear Systems Theory IV (Sandwich)

Main analytical result: for a "Sandwich" (AKA "Cascade") system



where

$$f(t) = \int s(t-\tau) L_1(\tau)\, d\tau \qquad \text{(linear)}$$

$$g(t) = N(f(t)) \qquad \text{("static nonlinear")}$$

$$r(t) = \int g(t-\tau) L_2(\tau)\, d\tau \qquad \text{(linear)}$$

then the nth order Wiener kernel K_n is

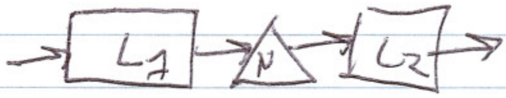
$$\boxed{K_n(\tau_1, \ldots, \tau_n) = a_n \int L_1(\tau_1 - s) L_1(\tau_2 - s) \cdots L_1(\tau_n - s) L_2(s)\, ds}$$

where a_n is the coefficient of h_n in the expansion of N, w.r.t. the power P that emerges from L_1.

i.e., if $P = \langle |f(t)|^2 \rangle$,

$$a_n = \frac{\int h_n(u)\, N(u)\, Gau(u)\, du}{P^n \cdot n!} \qquad \leftarrow \langle h_n(u)^2 \rangle$$

$$Gau(u) = \frac{1}{\sqrt{2\pi P}} e^{-u^2/2P}$$



$$K_1(\tau) = a_1 \int L_1(\tau - s) L_2(s)\,ds = a_1 (L_1 * L_2)(t)$$

$$K_2(\tau_1, \tau_2) = a_2 \int L_1(\tau_1 - s) L_1(\tau_2 - s) L_2(s)\,ds$$

Implications: ① K_1 is symmetric in L_1 + L_2, but K_2 is not

(nonlinear terms are sensitive to order of processing)

② Can test whether model structure is right
Generic K_2's are not of this form.
Easier in the frequency domain

define $\tilde{K}_2(\omega_1, \omega_2) = \iint e^{-i\omega_1\tau_1 - i\omega_2\tau_2} K_2(\tau_1, \tau_2)\,d\tau_1\,d\tau_2$

then $\tilde{K}_2(\omega_1, \omega_2) = \tilde{L}_1(\omega_1)\tilde{L}_1(\omega_2)\tilde{L}_2(\omega_1 + \omega_2)$ ⇒ ⓐ no isolated 0's
ⓑ simple structure for $\log K_2$
ⓒ K_2 predicts K_1

where $\tilde{L}_j(\omega) = \int e^{-i\omega\tau} L_j(\tau)\,d\tau$

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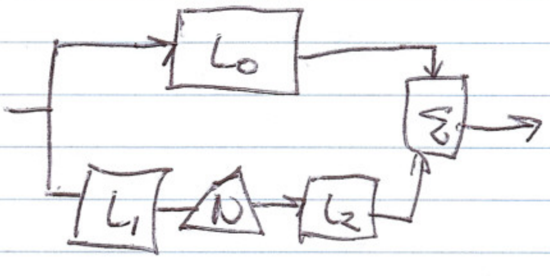
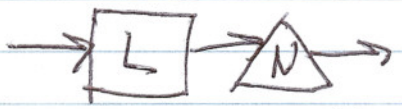
③ Easy generalization:



L_0 only affects K_1, not $K_2, K_3, \ldots$

Important special case



("Wiener" system)

[cf.

("Hammerstein system")]

"neuron cascade"

$$K_n(\tau_1,\dots,\tau_n) = a_n L_1(\tau_1)\cdots L_1(\tau_n)$$

Easy to tell if you have this form: K_1 predicts $K_2,\dots,K_n,\dots$

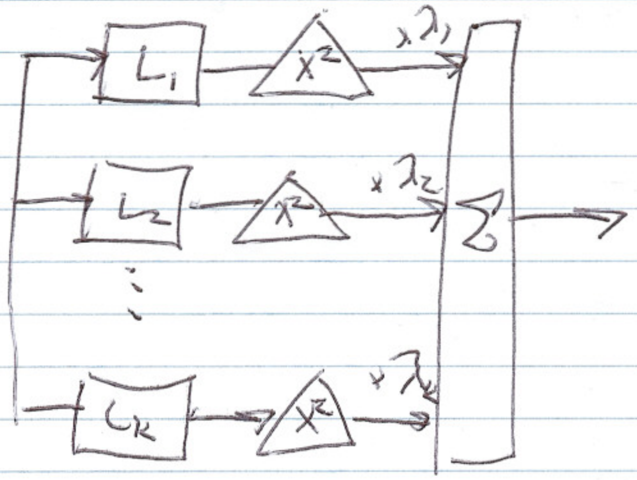
Connection with "spike triggered covariance" methods, etc.

Can take any $K_2(\tau_1,\tau_2)$ and attempt to write it as

$$K_2(\tau_1,\tau_2) = \sum_k \lambda_k f_k(\tau_1) f_k(\tau_2).$$

K_2 is symmetric, so this decomposition is guaranteed, - the f_k's are the eigenvectors of the matrix K_2.

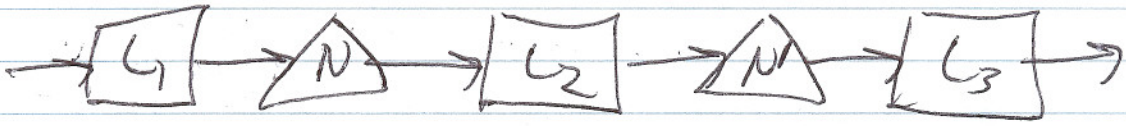
This corresponds to finding a system with the same 2^{nd}-order kernel, + the structure



Do the L_k's have "meaning"?

(No, such an expansion is guaranteed.)

The $L_1 N L_2$ model can fit well even if the internal structure is different
(good news – bad news)



If L_2 is $\sim I$, then NN' collapses

$$N'(N(f)) = N''(f)$$

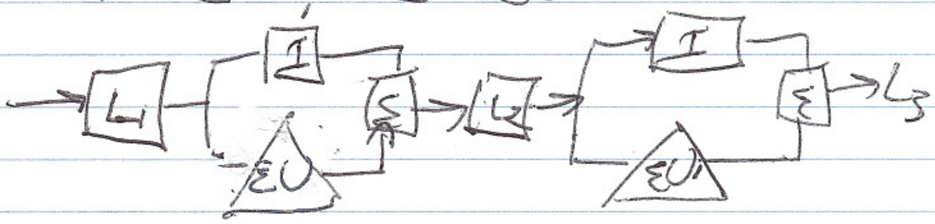
If N is $\sim I$, then L_1 & L_2 collapse.

If N' is $\sim I$, then L_2 & L_3 collapse.

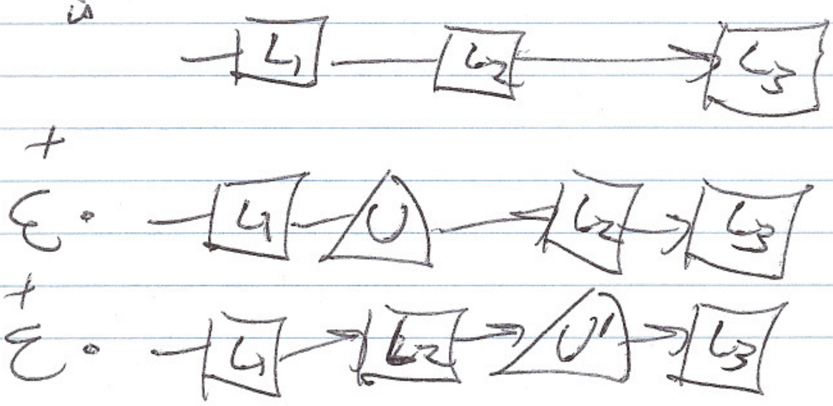
And if N & N' are both represented by, e.g.,

$$I + \varepsilon U, \quad I + \varepsilon U'$$

Then



is



i.e., a parallel sum of sandwiches.

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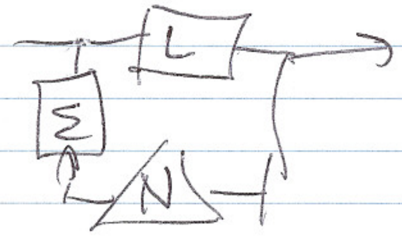
This idea also applies to nonlinear feedback



is exactly L

$+$

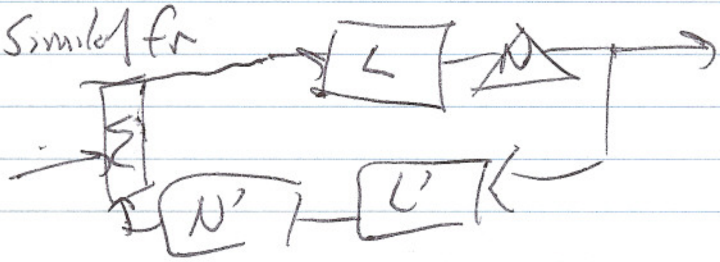
$L \to N \to L$

$+ L \to N \to L \to N \to L$

$+$

so if N is weak, a parallel sandwich will be a good approximation.

Similarly for



etc.

So an LNL- type model will often be a good description,

and a superposition of them is guaranteed to be a good description (with enough parallel paths)

and LN's can always be used to simulate an LNL!

Proof of the "sandwich theorem".

We need to project $\to\boxed{L_1}\to N \to\boxed{L_2}\to$ into the space spanned by the n^{th}-order functionals.

It suffices to do this for $\to\boxed{L_1}\to N\to$, since the

L_1NL_2-system is a superposition of lagged L_1N-systems.

(Superposition preserves the projection b/o linearity;

lagging preserves the projection b/o time-translation symmetry.)

We'll do this by showing that $\to\boxed{L}\to H_n\to$

is in the n^{th} orthogonal subspace, where H_n is a nonlinearity

corr. to h_n, the n^{th} Hermite
w.r.t. the power passed by L.

To do this, we need to show that A LH_n is orthogonal to $L'H_m$,

for any L', and any H_m. And we need to show that $L'H_m$

span the m^{th} subspace (for all choices of L').

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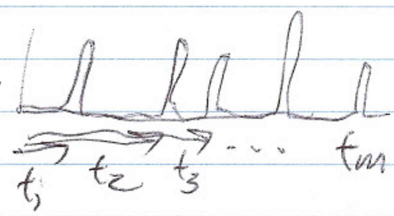
(B) To show that $L'H_m$ span the m^{th} subspace:

We need to construct a system whose response at time t is given by $s(t-t_1)\,s(t-t_2)\cdots s(t-t_m) + (\text{lower-order})$ by adding various $L'H_m$'s.

An $L'H_m$ might look like this: $L'(t) =$



So $L'H_m$ begins with

$$\left[s(t-t_1) + s(t-t_2) + \cdots + s(t-t_m)\right]^m, + \text{lower-order terms.}$$

This contains the requisite cross term $s(t-t_1)\cdots s(t-t_m)$ but it also contains "nuisance" m^{th}-order terms, like

$$s(t-t_1)^2\, s(t-t_3)\, s(t-t_4) \cdots s(t-t_m), \text{ etc.}$$

To kill that nuisance term, we could subtract off an $L''H_m$-term with $L''(t)$ having peaks at $t_1, t_3, \ldots, t_m$ [but not t_2].

This would still leave other nuisance terms, but fewer.

Eventually, we could subtract off all of them, using L's that were concentrated on subsets of $t_1, \ldots, t_m$.

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A (the hard part)

Say $L(s) = \sum a_k s_k$

$L'(s) = \sum b_k s_k$

$L(s)$ is Gaussian - because it is the sum of independent Gaussians - and has variance P

$L'(s)$ is also Gaussian - & has some other variance P'

$L(s)$ & $L'(s)$ may be correlated (since they both act on s);

say the correlation is ρ. i.e, covariance of $L(s)$ & $L'(s)$ is

$$\rho\sqrt{PP'} = \sum a_k b_k \langle s_k^2 \rangle = \left(\sum a_k b_k \frac{P}{\Delta t} \right)$$

We can replace the average $\langle h_n(L(s)) \cdot h_m(L'(s)) \rangle_{\vec{s}}$

by $\langle h_n(u)\, h_m(v) \rangle$

for u & v Gaussian variables with

variances P, P' and covariance $\rho\sqrt{PP'}$.

So we've replaced an ∞-dim integral (over s)

by a 2-d integral (over u, v).

$\langle h_n(u) h_m(v) \rangle$.

$u = v$: This is $n! \rho^n$ ($n=m$), 0 otherwise (orthogonality of the h's)

Indep ($\rho = 0$) $\langle uv \rangle = 0$: This is 0. Not obvious -- but it is an easy generating-function calculation.

(Partial) dependence $\langle h_n(u) h_m(v) \rangle = \begin{cases} n! \left(\rho\sqrt{PP'}\right)^n, & n = m \\ 0, & \text{otherwise.} \end{cases}$

This is "Price's Thm"

The "trick" is to write $u = ax + by$, $v = cx + dy$

with x, y unit-variance independent random Gaussians

$$\left.\begin{aligned} \langle u^2 \rangle &= a^2 + b^2 \\ \langle v^2 \rangle &= c^2 + d^2 \\ \langle uv \rangle &= ac + bd \end{aligned}\right\} \text{ easy to find } \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

indeed, $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$

+ then do the integrals over x & y.

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Can we get, directly, to

$$\tilde{K}_n(\omega_1, \ldots, \omega_n) = \hat{L}_1(\omega_1)\hat{L}_1(\omega_2) \ldots \hat{L}_1(\omega_n)\hat{L}_2(\omega_1 + \cdots + \omega_n)?$$

Sketch: Frequency-domain characterization of Wiener kernels.

Say $s(t) = \frac{1}{2}\sum\left(\alpha_k e^{i\omega_k t} + \bar{\alpha}_k e^{-i\omega_k t}\right)$

Many ω_k's, random α_k's.

A 0^{th} order system produces a constant

A linear system can produce $e^{i\omega_k t} + e^{-i\omega_k t}$

A quadratic system can produce
$e^{i(\omega_k + \omega_l)t}$
$e^{i(\omega_k - \omega_l)t}$
$e^{i(-\omega_k + \omega_l)t}$
$e^{-(i\omega_k + \omega_l)t}$

including the 0-freq.

The 0-freq component is the part that is non-orthogonal to the 0^{th} order part.

(Consider $s(t)^2$.)

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A cubic system will produce terms like

$$e^{i(\pm\omega_k \pm \omega_l \pm \omega_m)}$$

including terms with freq's $3\omega_k$, $2\omega_k \pm \omega_l$

that can't be made at low order

AND

terms with freq $\omega_k = \omega_k + \omega_k - \omega_k$

that can be produced at order 1. [the non-ortho piece]

So the n^{th} order part of the response to $R(s)$

that stays in the n^{th} order orthogonal subspace

is precisely the frequencies

$$\omega_{k_1} \pm \omega_{k_2} \pm \cdots \pm \omega_{k_n}$$

that cannot be made by smaller sums

Now let N be approximated by a polynomial, & use

trig identities. This leads directly to $\#$.